Register Number:

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ST. JOSEPH'S COLLEGE (AUTONOMOUS), BANGALORE-27

B.Sc. Mathematics - V SEMESTER

SEMESTER EXAMINATION: NOVEMBER 2020

MT 5118 – MATHEMATICS-V

Time: $2\frac{1}{2}$ hrs **Max Marks: 70**

This paper contains TWO printed pages and THREE parts

I Answer any FIVE of the following. **5 X 2 = 10**

1. In a commutative ring R prove that $a \cdot 0 = 0$ and $a(-b) = -ab, \ \forall a, b \in R$.
2. Give an example of an infinite noncommutative ring without unity.
3. In the ring $\mathbb{Z}_n[i] = \{a + bi \mid a, b \in \mathbb{Z}_n\}$, if n divides $a^2 + b^2$, for some $a, b \in \mathbb{Z}_n$, then show that $\mathbb{Z}_n[i]$ has zero-divisors.
4. Prove that the ideal $6\mathbb{Z}$ is neither prime nor maximal in the ring $\mathbb{Z}$.
5. Define ring homomorphism and kernel of homomorphism.
6. Find a_0 in the Fourier series expansion of $f(x) = x^2 \cos x$ in the interval $(0, 2\pi)$.
7. Evaluate using improper integrals: $\int_0^1 x^4(1-x)^7\,dx$
8. Evaluate: $\Gamma\left(-\frac{5}{2}\right)$

II Answer any SEVEN of the following. **7 X 6 = 42**

9. Construct the Cayley's tables for addition and multiplication of the elements in the ring $\mathbb{Z}_2[i] = \{a + ib \mid a, b \in \mathbb{Z}_2\}$. Solve the equation $x^2 - x - 1 = i$ in the ring $\mathbb{Z}_2[i]$.
10. Show that $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \{a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{6} \mid a_i \in \mathbb{Q}\}$ is a sub ring of the ring $\mathbb{R}$.

    Also prove that $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.

11. Define the integral domain and the field. Prove that a finite integral domain is a field.

12. (a) Let R be a commutative ring of characteristic 2. Prove that the set of all idempotent elements in R, form a subring.

(b) Prove that zero is the only nilpotent element in the ring of integers $\mathbb{Z}$. **(3+3)**

13. Let $A = < 2 - i >$ be an ideal of the ring $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ of Gaussian integers. Show that the quotient ring $\mathbb{Z}[i]/A$ is equivalent to $\mathbf{Z}_5$.

14. (a) Define a prime ideal and a maximal ideal.

(b) Let R be commutative ring with unity and A be an ideal of R. Prove that the quotient ring R/A is an integral domain if and only if A is a prime ideal. **(2+4)**

15. Show that the mapping $f: \mathbb{Z}_5 \longrightarrow \mathbb{Z}_{30}$, defined by $f(x) = 6x(mod 30)$ is a ring homomorphism. Find $\ker(f)$.

16. Let φ be a ring homomorphism from a ring R to a ring S.

(i) If B is an ideal of S, then prove that $\varphi^{-1}(B)$ is an ideal of R. Hence show that $\ker(\varphi)$ is an ideal of R.

(ii) If $1 \in R$, φ is onto and $S \neq \{0\}$, then prove that $\varphi(1) = 1'$ is the unity of S. **(4+2)**

17. Let $\varphi: \mathbb{Z}[x] \longrightarrow \mathbb{Z}$ be defined by $\varphi(p(x)) = p(0), \forall\, p(x) \in \mathbb{Z}[x]$.

(i) Prove that φ is an epimorphism.

(ii) Using Fundamental Theorem of Homomorphism, prove that the ideal $< x >$ is prime but not maximal in the ring $\mathbb{Z}[x]$. **(3+3)**

III Answer any THREE of the following. 3 X 6 = 18

18. Obtain Fourier series of the function $f(x) = x + x^2$ in the interval $(-\pi, \pi)$.

19. Obtain the Fourier cosine series of the function $f(x) = x\sin x$ in $(0, \pi)$.

20. Prove: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

21. Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

22. Prove that $\dfrac{\Gamma(n)\,\Gamma(n+\frac{1}{2})}{\Gamma(2n)} = \dfrac{\sqrt{\pi}}{2^{2n-1}}$.

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