Date:

Registration number:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
MSc. MATHEMATICS - IV SEMESTER
END-SEMESTER EXAMINATION: APRIL 2022
(Examination conducted in July-2022)

MTDE0918-BASIC OPERATOR THEORY

Time- 2.30 hrs Max Marks-70

This question paper contains **ONE** printed page and **ONE** part.

Answer any 7 questions 7 X 10 = 70

1. Show that T is a surjective linear isometry if the map $T: l^p(n)' \to l^q(n)$ is defined by $T(F) = \big(f(e_1), f(e_2), \dots, f(e_n)\big), where\ f \in (l^p)', e_j \in \mathbb{K}$ such that $e_j(i) = \delta_{ij}\ for\ i, j = 1,2,\dots,n.$ [10]
2. State and Prove Schur's lemma. [10]
3. Let X be a normed linear space and $A: X \to X$ be a compact operator. Then show that $\sigma_{eig}(A)$ is a countable subset of $\mathbb{K}$ and zero is the only possible limit point of $\sigma_{eig}(A)$. [10]
4. State and Prove spectral mapping theorem. [10]
5. a) Let $X\ and\ Y$ be normed linear spaces and $T: X \to Y$ be a surjective linear isometry then show that $T': Y' \to X'$ is a surjective linear isometry. [5]
   b) Let $X\ and\ Y$ be Hilbert spaces and $\Psi: X \times Y \to \mathbb{K}$ be a bounded sesquilinear functional. Prove that there exists a unique bounded linear operator $A: X \to Y$ such that $\Psi(x, y) =< Ax, y >$ for all $(x, y) \in X \times Y$. [5]
6. Let X be a Hilbert space over $\mathbb{C}$ and $A \in \mathfrak{B}(X)$ such that $< Ax, x > \in R\ for\ all\ x \in X$. Then, prove that A is self adjoint operator. [10]
7. Let X be a Hilbert space over $\mathbb{C}$ and $A \in \mathfrak{B}(X)$.Then, show that
   i) A is normal iff $||Ax|| = ||A^*x||$ for every $x \in X$ [5]
   ii) A is unitary iff A is surjective and $||Ax|| = ||x||$ for every $x \in X$ [5]
8. If $A \in \mathfrak{B}(X)$ is a self-adjoint operator, then show the following results
   i) $\{\alpha_A, \beta_A\} \subseteq \sigma(A) \subseteq [\alpha_A, \beta_A]$ [5]
   ii) $r_\sigma(A) = ||A|| = r_w(A) = max\{|\alpha_A|, |\beta_A|\}$ [5]
9. Let A be a compact self adjoint operator on X and $\{\lambda_i: i \in \Delta\}$ be the set of all non-zero eigen values of A. For each $i \in \Delta$ let $\{u_1^{(i)}, u_2^{(i)}, \dots, u_{m\,i}^{(i)}\}$ be an orthonormal basis of $N(A - \lambda_i I)$. Then $Ax = \sum_{i\in\Delta} \sum_j^{m_i} \lambda_i < x, u_j^i > u_j^i$, $\forall\ x \in X$ and $\cup\ i \in \Delta\{u_1^{(i)}, u_2^{(i)}, \dots, u_{m\,i}^{(i)}\}$ is an orthonormal basis of $N(A)^\perp$ when A is of infinite rank then show that $||A - \sum_{j=1}^n \lambda_i P_i|| \le \underbrace{max}_{i>n} |\lambda_i| \to 0\ as\ n \to \infty$. [10]
10. a) Let X be a separable Hilbert space, show that every Hilbert-Schmidt operator on X is a compact operator. [5]
    b) Let X, Y be Hilbert spaces and $T: X \to Y$ be a compact operator. Let $\{(\sigma_n, u_n, v_n): n \in \Delta\}$ be a singular system of T. Then, show that
    i) $\{u_n: n \in \Delta\}$ is an orthonormal basis for $N(T)^\perp$
    ii) $\{v_n: n \in \Delta\}$ is an orthonormal basis for $\overline{R(T)}$. [5]

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