Register Number:

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ST. JOSEPH'S COLLEGE (AUTONOMOUS), BANGALORE-27
B.Sc. MATHEMATICS - VI SEMESTER
SEMESTER EXAMINATION: APRIL 2022
(Examination conducted in JULY 2022)
MT 6118 – MATHEMATICS VII

Time: $2\frac{1}{2}$ hrs **Max Marks: 70**

This paper contains TWO printed pages and THREE parts.

I Answer any FIVE of the following. **5 X 2 = 10**

1. Given $(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2-b_2)$ and $c(a_1,a_2)=(ca_1,ca_2), \forall(a_1,a_2),(b_1,b_2)\in\mathbb{R}^2, \forall c\in\mathbb{R}$. Is $\mathbb{R}^2$ a vector space over $\mathbb{R}$ under these operations? Justify your answer.

2. (i) Define the subspace of a vector space.
   (ii) State the necessary and sufficient conditions for a subset to be a subspace.

3. Find the linear transformation $T:\mathbb{R}^3\to\mathbb{R}^2$ given that $T(1,0,0)=(-1,0)$, $T(0,1,0)=(1,1)$ and $T(0,0,1)=(0,-1)$.

4. Find the null space and nullity of the linear transformation $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T(x,y,z)=(x-y,2z)$.

5. Solve: $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$.

6. Form the partial differential equation by eliminating the arbitrary constants a and b from $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.

7. Solve the differential equation $\sqrt{p}+\sqrt{q}=1$.

8. Find the particular integral of the partial differential equation $(D^2-2DD'+D'^2)z=e^{x+2y}$.

II Answer any SEVEN of the following. **7 X 6 = 42**

9. (i) In a vector space $V(F)$ prove the following properties:
   (a) $0\cdot x=o,\ \forall x\in V,\ 0\in F,\ o\in V$
   (b) $(-a)x=-(ax),\ \forall x\in V,\ \forall a\in F$
   (c) $a\cdot o=o,\ \forall a\in F, o\in V$

   (ii) Show that the set *W* of all solutions of the linear homogeneous differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=0$ is a subspace of the vector space $C(\mathbb{R})$ of all continuous real valued functions. **(3+3)**

10. (i) Define the span of a subset S of a vector space V(F).
    (ii) Prove that the span of S is a subspace of V and it is contained in every subspace W of V that contains S. **(1+5)**

11. Prove that a finite set $S=\{u_1,u_2,\cdots,u_n\}$ of vectors in a vector space V(F) is linearly dependent if and only if $u_1=o$ or $u_{k+1}\in span(\{u_1,u_2,\cdots,u_k\})$ for some $k(1\le k<n)$.

12. (i) Define a basis for a vector space. Write the standard basis for the vector space $P_n(\mathbb{R})$ of all real polynomials of degree utmost n.
    (ii) Find a basis and the dimension of the subspace spanned by the subset $S=\left\{\begin{pmatrix}1 & -5\\ -4 & 2\end{pmatrix}, \begin{pmatrix}1 & 1\\ -1 & 5\end{pmatrix}, \begin{pmatrix}2 & -4\\ -5 & 7\end{pmatrix}, \begin{pmatrix}1 & -7\\ -5 & 1\end{pmatrix}\right\}$ of the vector space $M_2(\mathbb{R})$ of all real square matrices of order 2. **(2+4)**

13. State and prove the Dimension Theorem (Rank-Nullity Theorem).

14. Find the range space, null space, rank, and nullity of the linear transformation $T:\mathbb{R}^2\to\mathbb{R}^3$ defined by $T(a_1,a_2)=(a_1+a_2,0,2a_1-a_2)$. Also verify the Rank-Nullity theorem.

15. Prove that a linear transformation $T:V(F)\to W(F)$ is one to one if and only if *T* carries linearly independent subsets of V onto linearly independent subsets of W.

16. Find the matrix $[T]_\alpha^\beta$ of the linear transformation $T:\mathbb{R}^2\to\mathbb{R}^3$ defined by $T(x,y)=(x+y,x,3x-y)$ relative to bases $\alpha=\{(1,1),(3,1)\}$ and $\beta=\{(1,1,1),(1,1,0),(1,0,0)\}$.

17. Let $T:P_1(\mathbb{R})\to P_1(\mathbb{R})$ be a linear operator defined by $T(p(x))=p'(x)$, where $p'(x)$ is the derivative of $p(x)$. Let $B_1=\{1,x\}$ and $B_2=\{1+x,1-x\}$ be the ordered bases for $P_1(\mathbb{R})$.
    (i) Find the change of coordinate matrix Q that changes B_2 coordinates into B_1 coordinates and find Q^{-1}.
    (ii) Compute $[T]_{B_1}$.
    (iii) Find $[T]_{B_2}$ using $[T]_{B_1}, Q \text{ and } Q^{-1}$. **(2 + 2 + 2)**

III Answer any THREE of the following. 3 X 6 = 18

18. Verify the condition for integrability and solve: $(y^2+z^2-x^2)\,dx-2xy\,dy-2xz\,dz=0$.

19. Solve: $(y-z)p+(z-x)q=x-y$.

20. Form the partial differential equation by eliminating the arbitrary functions f and g in $z=\frac{1}{y}[f(x+ay)+g(x-ay)]$.

21. Solve by Charpit's method: $z^2(p^2+q^2+1)=1$.

22. Solve: $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\partial y}+4\frac{\partial^2 z}{\partial y^2}=\sin(4x+y)$.