**ST. JOSEPH'S UNIVERSITY, BENGALURU -27**
**M.Sc. (MATHEMATICS) – I SEMESTER**
**SEMESTER EXAMINATION: NOVEMBER 2022**
**(Examination conducted in December 2022)**
**MT7221 – MATHEMATICS**

| Registration Number: |
|---|
| Date & session: |

**Instructions:**
- **All the questions carry equal Marks.**
- **All correct options for the Multiple-Choice Questions should be written in the answer booklet.**

This question paper contains **TWO** printed pages and **ONE** part

**Time - 2 hrs** **Max Marks - 50**

**Answer any 5 questions**

1. a) Let a function $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Show that $f \in \mathcal{R}[a,b]$.
   b) Find the upper and lower integral for the function $f$ defined on $[0,2]$ by
   $$f(x) = \begin{cases} x^2 & if\ x\ is\ rational \\ x^3 & if\ x\ irrational \end{cases}$$ [5+5]
2. a) Let a function $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and $f$ be continuous on $[a,b]$except for a finite number of points in $[a,b]$. Prove that $f \in \mathcal{R}[a,b]$.
   b) Using first mean value theorem, show that $\frac{1}{3\sqrt{2}} < \int_0^1 \frac{x^2}{\sqrt{1+x}} < \frac{1}{3}$. [5+5]
3. a) Show that the sequence $\{f_n\}$ converges uniformly on $[0,1]$ where $f_n(x) = \frac{x}{1+nx^2}$ , $x \in [0,1]$ .
   b) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real $x$. [6+4]
4. a) Show that any subset of a countable set is countable.
   b) If $A$ and $B$ are countable sets then show that $A \times B$ is countable. [5+5]
5. a) Show that on the Euclidean plane $\mathbb{R}^2$ the function $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ is metric on $\mathbb{R}^2$.
   b) Show that $(\mathbb{R}^m, d)$ is complete where $d$ denotes the Euclidean metric . [5+5]
6. a) Let $f: X \to Y$ be a function where $X$ and $Y$ are metric spaces. Show that $f$ is continuous iff for every subset $B \subseteq Y\ then\ f^{-1}(Int\ (B)) \subseteq Int(f^{-1}(B))$.
   b) Prove that the function $f:(0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous. [6+4]
7. a) Let $(X, d)$ be a metric space. Prove the following properties:
   i) $\emptyset$ and $X$ are open
   ii) Arbitrary union of open sets is open
   iii) Finite intersection of open sets is open.
   b) Let $A$ be a non-empty proper closed subset of $\mathbb{R}$. Then $A$ is [5+1+1+1+1+1]
   i) the closure of the interior of $A$
   ii) a countable set
   iii) a compact set

iv) not open

c) If $f(x)$ is defined on $(0,2)$ by $f(x) = \begin{cases} x + x^2 \text{ when } x \text{ is rational} \\ x^2 + x^3 \text{ when } x \text{ is irrational} \end{cases}$. The value of the upper Riemann integral in $(0,2)$.

i) $\frac{12}{83}$ ii) $\frac{83}{12}$ iii) $\frac{53}{12}$ iv) $\frac{123}{83}$

c) If a convergent sequence in a metric space has infinitely many distinct terms, then its limit is the set of terms of the sequence of

i) Isolated point
ii) Limit points
iii) Interior points
iv) Exterior points

e) For the series $\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}$ on the interval $[0,1]$ which of the following is/are true?

i) Converges pointwise but not uniformly
ii) Doesn't converge pointwise
iii) Converges uniformly
iv) Converges pointwise to a discontinuous function.

f) Let the function $f_n(x) = x^n, \ \forall\, x \in [0,1)$. Then the sequence $\{f_n\}$ is,

i) uniformly convergent
ii) pointwise convergent but not uniformly convergent
iii) pointwise and uniformly convergent
iv) uniform but not pointwise convergent.