Register Number:

Date:

ST. JOSEPH'S COLLEGE, (Autonomous) BENGALURU-27
M.SC MATHEMATICS - III SEMESTER
SEMESTER EXAMINATION: OCTOBER, 2022
(Examination conducted in December 2022)

MT 9122: Functional Analysis

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains **TWO** printed pages and ONE part.
2. Attempt any **SEVEN FULL** questions.
3. All multiple choice questions have **one or more** correct options. Write **all** the correct options in your answer booklet.

*** BEST WISHES ***

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1. a) Show that $L^\infty(E) \subset L^p(E)$ for all $1 \leq p < \infty$, where $m(E) < \infty$. Show with an example that the inclusion doesn't hold if we drop the finite measure condition on E. [7 m]

   b) Which of the following is a Banach space. [3 m]

   I) $C([0,1])$ the set of all continuous function on $[0,1]$ with $|| \, ||_\infty$.

   II) $C([0,1])$ the set of all continuous function on $[0,1]$ with $|| \, ||_p$.

   III) $P([0,1])$ the set of all polynomial function on $[0,1]$ with $|| \, ||_\infty$.

   IV) $P([0,1])$ the set of all polynomial function on $[0,1]$ with $|| \, ||_p$.

2. a) Show that any two norms on a finite dimensional space are equivalent. [6 m]

   b) Choose all the correct statements. [4 m]

   I) The norms $||x|| = ||x||_\infty + ||x'||_\infty$ and $||x||_0 = |x(a)| + ||x'||_\infty$ on $C^1([a,b])$ are equivalent.

   II) The norms $||x|| = ||x||_\infty + ||x'||_\infty$ and $||x||_0 = |x(a)| + ||x'||_\infty$ on $C^1([a,b])$ are not equivalent.

   III) The norms $||x||_1 = \int_0^1 |x(t)|dt$ and $||x||_\infty = \sup\{|x(t)| : t \in [a,b]\}$ on $C([a,b])$ are equivalent.

   IV) The norm $||x||_\infty = sup\{|x(t)| : t \in [a,b]\}$ is stronger than $||x||_1 = \int_0^1 |x(t)|dt$ on $C([a,b])$.

3. a) State and prove Riesz lemma. [6 m]

   b) Choose all the correct statements. [4 m]

   I) The space $C[0,1]$ is a closed subspace of $L^\infty(0,1)$ with sup-norm.

   II) The space $C[0,1]$ is a closed subspace of $L^1(0,1)$ with $\| \cdot \|_1$.

   III) The space c_{00} is a closed subspace of c_0 with sup-norm.

   IV) The space c_{00} is dense in c_0 with sup-norm.

4. a) Define Fredholm operator and show that it is continuous. [7 m]

   b) Let (X, S, μ) be a measure space. Choose all the correct statements. [3 m]

   I) Let $g \in L^\infty(\mu)$. Let $T(f) = gf$. Then $T \in B(L^1(\mu), L^1(\mu))$.

   II) Let $g \in L^\infty(\mu)$. Let $T(f) = gf$. Then $T \in B(L^2(\mu), L^1(\mu))$.

   III) Let $g \in L^2(\mu)$. Let $T(f) = gf$. Then $T \in B(L^2(\mu), L^1(\mu))$.

5. a) Let $X = (C^1[0,1], \| \cdot \|_*)$, where $\|x\|_* = \|x\|_\infty + \|x'\|_\infty$ and $Y = (C[0,1], \| \cdot \|_\infty)$. Show that the linear operator $A : X \to Y$ defined by $A(x) = x'$ is continuous and find $||A||$. [6 m]

   b) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 & 5 \\ -1 & 0 & 2 & -1 & 1 \\ 3 & 0 & 0 & 2 & -5 \\ 4 & 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & -2 & -1 \end{bmatrix}.$$

   The matrix A defines a linear transformation $T_A : \mathbb{K}^5 \to \mathbb{K}^5$ is a bounded operator on [4 m]

I) $(\mathbb{K}^5, \|\cdot\|_1)$ with norm 13.
II) $(\mathbb{K}^5, \|\cdot\|_1)$ with norm 15.
III) $(\mathbb{K}^5, \|\cdot\|_\infty)$ with norm 15.
IV) $(\mathbb{K}^5, \|\cdot\|_\infty)$ with norm 13.

6. a) Prove the projection theorem: "Let H be a Hilbert(complete inner product) space and F be a closed subspace of H. Then, $H = F \oplus F^\perp$ and $(F^\perp)^\perp = F$." [5 m]

   b) Give an example to show that the completeness property is necessary for the conclusion of the Projection theorem. [5 m]

7. a) State and prove Riesz Representation Theorem. [7 m]

   b) Which of the following represent an orthogonal projection in $L^2(-\pi, \pi)$. [3 m]

   I) T defined by $Tf(t) = \frac{1}{2}(f(t) + f(-t))$ for all $f \in L^2(-\pi, \pi)$
   II) T defined by $Tf(t) = tf(t)$ for all $f \in L^2(-\pi, \pi)$
   III) T defined by $Tf(t) = \chi_{[0,\pi]}(t)f(t)$ for all $f \in L^2(-\pi, \pi)$.

8. a) Let V be a normed linear space, W be a subspace of V and X be a finite dimensional normed linear space. Show that any continuous linear functional from $T : W \to X$ can be extended to all of V. [7 m]

   b) Choose all the correct statements. [3 m]

   I) Let $x \in \ell^5$. There exists $y \in \ell^{\frac{5}{4}}$ such that $\|y\|_{\frac{5}{4}} = 1$ and $\|x\|_{\frac{5}{4}} = \left|\sum_{i=1}^{\infty} x_i y_i\right|$

   II) Let $x \in \ell^{\frac{5}{4}}$. There exists $y \in \ell^5$ such that $\|y\|_5 = 1$ and $\|x\|_{\frac{5}{4}} = \left|\sum_{i=1}^{\infty} x_i y_i\right|$

   III) Let $x = (x_i) \in \ell_\infty$, where $x_i = 1 - \frac{1}{i}$. Then, $\|x\|_\infty = \left|\sum_{i=1}^{\infty} x_i y_i\right|$.

9. a) State and prove Hahn Banach Separation theorem. [7 m]

   b) Choose all the correct statements. [3 m]

   I) Let $V = C_c(\mathbb{R})$ with the sup-norm. The set $H = \left\{ f \in V \,\middle|\, \int_{-\infty}^{\infty} f(t)dt = 1 \right\}$ is a closed hyperplane in V

   II) Let $x \in \ell^4$. The set $H = \left\{ y \in \ell_{\frac{4}{3}} \,\middle|\, \sum_{i=1}^{\infty} x_i y_i = 1 \right\}$ is a closed hyperplane in $\ell^{\frac{4}{3}}$

   III) Let $V = C([0, 1])$ with the sup-norm. The set $H = \left\{ f \in V \,\middle|\, \int_0^1 g(t)f(t)dt = 1 \right\}$ is a closed hyperplane in V.

10. a) State and prove closed graph theorem. [7 m]

   b) Let V be a Banach space and let W be a non-zero,closed and proper subspace. Define $\pi : V \to V/W$ by $\pi(x) = x+W$. Which of the following statements are true? [3 m]

   I) The map π is injective
   II) The map π is surjective
   III) The map π is an open map.

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