Register number:

Date and session:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.Sc (MATHEMATICS) - IV SEMESTER
SEMESTER EXAMINATION: APRIL, 2023
(Examination conducted in May 2023)

MTDE 0522: Differential Geometry

(For current batch students only)

Duration: $2\frac{1}{2}$ Hours **Max. Marks:** 70

1. The paper contains two printed pages and one part.
2. Answer any **SEVEN FULL** questions.

1. a) Let $V = y^2U_1 - xU_3$ and let $f = xy$ and $g = z^3$. Compute the functions $V[fg]$. **[2m]**

   b) Let $v_{\mathbf{p}}$ be the tangent vector to $\mathbb{R}^3$ with $v = (2, -1, 3)$ and $\mathbf{p} = (2, 0, -1)$ and let $f = e^x \cos y$. Compute $v_{\mathbf{p}}[f]$. **[3m]**

   c) Which of the following are 1-forms? In each case ϕ is the function on tangent vectors such that the value of ϕ on (v_1, v_2, v_3) is given by **[5m]**

   i) $v_1 - v_3$ ii) 2 iii) $p_1 - p_3$.

2. a) Show that if α is a regular curve in $\mathbb{R}^3$, then there exists a reparametrization β of α such that β has unit speed. **[5m]**

   b) Find the unit-speed reparametrization of the curve $\alpha(t) = (\cosh t, \sinh t, t)$ based at t=0. **[5m]**

3. a) Show that a regular curve α with $\kappa > 0$ is a cylindrical helix if the ratio τ/κ is a constant. **[5m]**

   b) Consider the curve $\beta(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}}\right)$ defined on $(-1, 1)$. Show that β has unit speed, and compute T, κ, N of the curve. **[5m]**

4. a) Show that if F is an isometry of $\mathbb{R}^3$, then there exists a unique translation T and a unique orthogonal transformation C such that $F = TC$. **[6m]**

   b) In each of the cases decide whether F is an isometry on $\mathbb{R}^3$. If so, find its translation and orthogonal parts. Here $\mathbf{p} = (p_1, p_2, p_3)$. **[4m]**

   (i) $F(\mathbf{p}) = -\mathbf{p}$ (ii) $F(\mathbf{p}) = (p_3 - 1, p_2 - 2, p_1 - 3)$ (iii) $F(\mathbf{p}) = (p_1, p_2, 0)$

5. a) Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping $\mathbf{x}(u, v) = (u + v, u - v, uv)$. Show that $\mathbf{x}$ is a proper patch. **[6m]**

   b) For which values of c is $M : z(z - 2) + xy = c$ a surface? **[4m]**

6. a) Show that the surface of revolution $M : (\sqrt{x^2 + y^2} - 4)^2 + z^2 = 4$ is a torus. Also, write a parametrization for this surface. **[5m]**

   b) Show that the geographical patch of the unit sphere is a one-one parametrization of the sphere except for the poles. **[5m]**

7. a) Let $\mathbf{x} : \mathbf{D} \to \mathbb{R}^3$ be a proper patch on an open subset $\mathbf{D}$ of $\mathbb{R}^2$. Pick the correct statement(s) from the options given below:

(i) $\mathbf{x}$ is a homeomorphism from $\mathbf{D}$ to $\mathbf{x}(\mathbf{D})$.

(ii) The Jacobian of $\mathbf{x}$ need not have rank 2 always.

(iii) $\mathbf{x}(\mathbf{D})$ is an example of a simple surface.

(iv) The Jacobian of $\mathbf{x}$ always has rank 2. **[4m]**

b) Define the quadratic approximation of a surface near a point. Find the quadratic approximation near the origin for the surface $M : z = \log(\cos x) - \log(\cos y)$. **[4m]**

c) Suppose $M \in \mathbb{R}^3$ is a surface and $\mathbf{p} \in M$. Let $S_{\mathbf{p}}$ denote the shape operator of M at the point $\mathbf{p}$. Suppose $\{v_{\mathbf{p}}, w_{\mathbf{p}}\}$ is a linearly independent subset of the tangent space $T_{\mathbf{p}}(M)$ and $H(\mathbf{p}) = 0$, where H denotes the mean curvature of M at $\mathbf{p}$. Let $K(\mathbf{p})$ denote the Gaussian curvature of M. Pick the correct statement(s) from the options given below.

(i) If $K(\mathbf{p}) = 0$, then $\mathbf{p}$ is an umbilic point of M.

(ii) If $K(\mathbf{p}) \neq 0$, then $\mathbf{p}$ is not an umbilic point of M.

(iii) If $K(\mathbf{p}) \neq 0$, then $K(\mathbf{p}) > 0$.

(iv) If $K(\mathbf{p}) \neq 0$, then $\{S_{\mathbf{p}}(v_{\mathbf{p}}), S_{\mathbf{p}}(w_{\mathbf{p}})\}$ is linearly independent. **[2m]**

8. a) Let M be a surface in $\mathbb{R}^3$. Prove that the shape operator at each point $\mathbf{p} \in M$ is a linear operator on the tangent space $T_{\mathbf{p}}(M)$. **[6m]**

b) Compute the shape operator of the sphere $\Sigma : \{(x, y, z) : x^2 + y^2 + z^2 = r\}$. **[4m]**

9. Let $M \subset \mathbb{R}^3$ be a surface and $\mathbf{p} \in M$. Define the principal curvatures and principal directions of M at $\mathbf{p}$. If S is the shape operator of M, prove that the principal curvatures of M are precisely the eigenvalues of S and the principal directions are the corresponding eigenvectors. **[10m]**

OR

Compute the Gaussian and Mean curvatures for the saddle surface $M : z = xy$. **[10m]**

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