Register number:

Date and session:

ST JOSEPH'S UNIVERSITY, BANGALORE - 27
M.Sc MATHEMATICS - II SEMESTER
SEMESTER EXAMINATION: APRIL 2024
(Examination conducted in May/June 2024)
MT 8121: ALGEBRA II
(For current batch students only)

Duration: 2 Hours **Max. Marks:** 50

1. The paper contains two printed pages and one part.
2. Answer any **FIVE FULL** questions.
3. All multiple choice questions have 1 or more than one correct option. Full marks will be awarded only for writing **all correct options** in your answer script.
4. All True/False questions must be justified.

1. a) Compute $(1+\theta+\theta^2)^{-1}$ in $\mathbb{Q}(\theta)$ where θ is a root of the irreducible polynomial x^3-2x-2.

   b) Let F be a field and $p(x) \in F[x]$ be irreducible. Suppose K is an extension over F containing a root α of $p(x)$ then show that $F(\alpha)$ is isomorphic to $\dfrac{F[x]}{\langle p(x)\rangle}$. [5+5]

2. a) Let K/F be a field extension and let $\alpha \in K$ be algebraic over F. Show that there exists a unique monic irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ which has α as a root. Further, show that given any polynomial $f(x) \in F[x]$, it has α as a root if and only if $m_{\alpha,F}(x)$ divides $f(x)$.

   b) Compute $\left[\mathbb{Q}\left(\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}\right):\mathbb{Q}\right]$. [6+4]

3. a) Compute the splitting field of x^4-4 over $\mathbb{Q}$ and hence compute its dimension.

   b) Show that every irreducible polynomial over a finite field F is separable. [5+5]

   OR

   c) Let F be the rational field $\mathbb{Z}_3(t)$. Is the polynomial x^3-t separable over F? Justify.

   d) Compute the cyclotomic polynomial $\Phi_{12}(x)$. [4+6]

4. a) Show that $\text{Aut}(\mathbb{Q}(\sqrt{2},i))$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

   b) Which of the following are Galois extensions? (Write all the correct options in your answer booklet. Explanation is not required)

      i. The splitting field of x^4-4 over $\mathbb{Q}$
      ii. The splitting field of x^3-t over $\mathbb{Z}_2(t)$
      iii. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$
      iv. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. [6+4]

5. a) State the fundamental theorem of Galois Theory.

b) Draw the 1-1 correspondence between subgroups of $\text{Gal}(\mathbb{F}_{2^{36}}/\mathbb{F}_2)$ and subfields of $\mathbb{F}_{2^{36}}$.

[4+6]

6. a) Prove that the irreducible polynomial $x^4 + 1 \in \mathbb{Z}[x]$ is reducible modulo every prime number p.

   b) Show that the Galois group of the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_n^{\times}$.

[5+5]

7. a) Compute the discriminant of the polynomial $x^4 - 1$.

   b) Match the polynomials given below to their corresponding Galois groups. (You can use the formula, discriminant D of $f(x) = x^3 + ax^2 + bx + c$ is $D = a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$ wherever required.)

Polynomial	Galois group
$x^3 + 2x + 14$	$\{1\}$
$x^3 + x^2 - x - 1$	$\mathbb{Z}_2$
$x^3 + x^2 - 2x - 1$	S_3
$x^3 + x^2 + x$	A_3

[4+6]