Register Number:

Date: 9-01-2020

St. Joseph's College (Autonomous), Bangalore
M.Sc Mathematics - I Semester
End Semester Examination: January, 2021
MT7318: Linear Algebra

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains four printed pages.
2. Attempt any **SEVEN FULL** questions.
3. In objective type questions, one or more options could be correct. Full marks will be awarded only if all the options are correctly marked. There is no partial marking for these questions.

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1. a) Let V and W be two vector spaces over $\mathbb{Q}$ and $T : V \to W$ be a function. Prove that T is additive (i.e., $T(v_1 + v_2) = T(v_1) + T(v_2)$, for every $v_1, v_2 \in V$) if and only if T is linear. **[7m]**

   b) The dimension of the vector space of all symmetric 6×6 matrices over a field F is
   (i) 30 (ii) 15 (iii) 21 (iv) 42. **[3m]**

2. a) Let $V(F)$ be a finite dimensional vector space and $T, S \in End(V)$. Then prove that $\text{rk}(TS) \leq \text{rk}(T)$ and $\text{rk}(TS) \leq \text{rk}(S)$. Further, if S is invertible, show that $\text{rk}(ST) = \text{rk}(TS) = \text{rk}(T)$. **[8m]**

   b) Suppose T is a linear map on a finite dimensional vector space $V(F)$ such that the constant term in the characteristic polynomial of T is 0. Pick the correct statement(s) from the options given below.
   (i) The constant term in the minimal polynomial of T could be nonzero.
   (ii) T is onto.
   (iii) Nullity$(T) > 0$.
   (iv) There exists a nonzero linear map S such that $TS = ST = 0$.
   [2m]

3. a) Define a nilpotent linear operator on a vector space $V(F)$. If T is a nilpotent linear operator on a finite dimensional vector space V, prove that 0 is the only eigen value of T. **[5m]**

   b) Prove that the 2×2 matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is not diagonalizable. **[3m]**

   c) Let V be the vector space of all infinitely differentiable real valued functions on $\mathbb{R}$. Let $T = \frac{d^2}{dx^2}$ be the linear map on V. Pick the correct statement(s) from the options given below.
   (i) The eigen space corresponding to each eigen value is one dimensional.
   (ii) The eigen space corresponding to each eigen value is two dimensional.

(iii) Every real number is an eigen value of T.

(iv) The nullity of T is 2.

[2m]

4. a) Let $V(F)$ be a finite dimensional vector space and $T \in End(V)$. Let λ be an eigen value of T. Define the algebraic and geometric multiplicities corresponding to λ. Prove that the geometric multiplicity of λ is less than or equal to its algebraic multiplicity. **[7m]**

   b) Suppose $V(F)$ is a 5 dimensional vector space and $T \in End(V)$. Pick the correct statement(s) from the options given below.

   (i) If the characteristic polynomial of T is $(-1)^5(x-2)^3(x-5)^2$, then the minimal polynomial of T has to be $(x-2)^3(x-5)^2$.

   (ii) If the minimal polynomial of T is $(x-2)^3(x-5)^2$, then T is not diagonalizable.

   (iii) If the minimal polynomial of T is $(x-2)^3(x-5)^2$, then the geometric multiplicity of both the eigen values is 1.

   (iv) If the minimal polynomial of T is $(x-2)^3(x-5)^2$, then the characteristic polynomial of T has to be $(-1)^5(x-2)^3(x-5)^2$.

[3m]

5. a) Let T be a linear map on a vector space $V(F)$. Then what do we mean by a T-invariant subspace of V. If $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, find an A-invariant subspace of $\mathbb{R}^3$. **[3m]**

   b) Suppose $V(F)$ is a 6 dimensional vector space $T \in End(V)$. Write the Jordan canonical form of T if

   (i) the minimal polynomial of T is $(x-2)^4(x-5)^2$.

   (ii) the minimal polynomial of T is $(x-2)^3(x-5)$, the algebraic multiplicity of the eigen value 2 is 4 and that of the eigen value 5 is 2. **[5m]**

   c) Let $V(F)$ be a finite dimensional vector space and W be a T-invariant subspace of V. Pick the correct statement(s) from the options given below.

   (i) W is $q(T)$-invariant, for every $q(x) \in F[x]$.

   (ii) If $\overline{T}$ is the linear map on the quotient space $\frac{V}{W}$ induced by T, then $\text{rk}(\overline{T}) \leq \text{rk}(T)$.

   (iii) The minimal polynomial of T always divides the minimal polynomial of $\overline{T}$.

   (iv) The minimal polynomial of $\overline{T}$ divides the minimal polynomial of T.

[2m]

6. a) Let $V = M_{m \times n}(\mathbb{R})$. Define a function $\langle\, , \,\rangle : V \times V \to \mathbb{R}$ by $\langle A, B\rangle = \text{trace}(B^t A)$, where B^t is the transpose of B. Prove that this function is an inner product on V. **[8m]**

   b) Let $V(F)$ be an inner product space. Then pick the correct statement(s) from the options given below.

   (i) $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$, only if either u or v is zero.

   (ii) $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$, for all $u, v \in V$.

   (iii) $\|\alpha v\| = \overline{\alpha}\|v\|$, for all $\alpha \in F$ and $v \in V$.

   (iv) $\|\cdot\|$ is a nonnegative real valued function on V always. **[2m]**

7. a) Consider the basis $\{v_1 = (1,1,1),\ v_2 = (0,1,1),\ v_3 = (0,0,1)\}$ of $\mathbb{R}^3$. Obtain the corresponding orthonormal basis of $\mathbb{R}^3$ by means of Gram-Scmidt orthogonalization. **[7m]**

b) Pick the correct statement(s) from the options given below:

(i) If $\{v_1, \ldots, v_r\}$ is a linearly independent set in an inner product space V, then there exists an orthonormal set $\{u_1, \ldots, u_r\} \subseteq V$ such that $L\{v_1, \ldots, v_r\} = L\{u_1, \ldots, u_r\}$.

(ii) If $\{v_1, \ldots, v_r\}$ is an orthogonal set in an inner product space V, then $\{v_1, \ldots, v_n\}$ is linearly independent.

(iii) If $\{v_1, \ldots, v_r\}$ is an orthonormal set in an inner product space V, then $\{v_1, \ldots, v_n\}$ is linearly independent.

(iv) If $V(F)$ is a finite dimensional inner product space, then W and $W^{\perp}$ are always of same dimension.

[3m]

8. a) Prove that a symmetric $n \times n$ matrix with real entries is always diagonalizable. **[8m]**

b) Pick the correct statement(s) from the options given below:

(i) Any nonzero linear functional is surjective.

(ii) Let $V = \mathcal{P}(\mathbb{R})$. Then $\int_0^x (\,)dx$ is an example of a linear functional on V.

(iii) Let V be an inner product space. Then, for each fixed $u \in V$, the map $T : V \to F$ defined by $T(v) = \langle u, v \rangle$ for all $v \in V$, is always linear, irrespective of the field considered.

(iv) Let V be an inner product space. Then, for each fixed $u \in V$, the map $T : V \to F$ defined by $T(v) = \langle v, u \rangle$ for all $v \in V$, is always linear, irrespective of the field considered.

[2m]

9. a) Let T be a positive definite linear operator on a finite dimensional inner product space V. Prove that all eigen values of T are positive. Hence deduce that if A is a positive definite matrix, then $\det(A)$ is positive. **[4m]**

b) Prove that if A is a positive definite matrix then A^{-1} is positive definite. **[4m]**

c) Pick the correct statement(s) from the options given below.

(i) The determinant of an orthogonal matrix is ± 1.

(ii) The determinant of a unitary matrix is ± 1.

(iii) If U is an orthogonal or a unitary operator on a finite dimensional inner product space V, then $\|Uv - Uw\| = \|v - w\|$, for all $v, w \in V$.

(iv) An orthogonal matrix can have zero as one of its rows.

[2m]

10. a) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then find the singular values of A. If $A = U\Sigma V^t$ is a singular value decomposition of A, then write the matrix Σ corresponding to this A. **[3m]**

b) Define a symmetric bilinear form and a quadratic form on a finite dimensional vector space $V(F)$. Find the symmetric matrix corresponding to the quadratic form
$q(x, y, z) = 3x^2 + 4xy - y^2 + 8xz - 6yz + z^2$. **[4m]**

7. a) Consider the basis $\{v_1 = (1,1,1),\ v_2 = (0,1,1),\ v_3 = (0,0,1)\}$ of $\mathbb{R}^3$. Obtain the corresponding orthonormal basis of $\mathbb{R}^3$ by means of Gram-Scmidt orthogonalization. **[7m]**

b) Pick the correct statement(s) from the options given below:

(i) If $\{v_1, \ldots, v_r\}$ is a linearly independent set in an inner product space V, then there exists an orthonormal set $\{u_1, \ldots, u_r\} \subseteq V$ such that $L\{v_1, \ldots, v_r\} = L\{u_1, \ldots, u_r\}$.

(ii) If $\{v_1, \ldots, v_r\}$ is an orthogonal set in an inner product space V, then $\{v_1, \ldots, v_n\}$ is linearly independent.

(iii) If $\{v_1, \ldots, v_r\}$ is an orthonormal set in an inner product space V, then $\{v_1, \ldots, v_n\}$ is linearly independent.

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(iv) Let V be an inner product space. Then, for each fixed $u \in V$, the map $T : V \to F$ defined by $T(v) = \langle v, u \rangle$ for all $v \in V$, is always linear, irrespective of the field considered.

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c) Pick the correct statement(s) from the options given below.

(i) If A be an $m \times n$ matrix over $\mathbb{R}$, then the singular values of A are always ≥ 0.

(ii) If A be an $m \times n$ matrix over $\mathbb{R}$, then $A^t A$ is always positive definite.

(iii) If A be an $n \times n$ matrix over $\mathbb{R}$, then the nullity$(A) = n-$ (number of nonzero singular values of A).

(iv) If A be an $n \times n$ matrix over $\mathbb{R}$, then A is invertible if and only if the number of nonzero singular values of A is equal to n. **[3m]**

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