Register Number:

Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.SC MATHEMATICS - III SEMESTER
SEMESTER EXAMINATION: OCTOBER, 2021
(Examination conducted in January-March 2022)
MTDE 9518: ALGEBRAIC TOPOLOGY

Duration: 2.5 Hours **Max. Marks:** 70

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1. The paper contains two pages.
2. Answer any **SEVEN FULL** questions.
3. All multiple choice questions have 1 or more than one correct option. Full marks will be awarded only for writing **all correct options** in your answer script.
4. All true or false questions must be justified.

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1. a) Show that the relation of path homotopy defines an equivalence relation on the set of all paths in X from x_0 to x_1. [6]
   b) True/False: If $A \cap B \neq \emptyset$ and A, B are contractible then so is $A \cup B$. [4]
2. a) Show that the operation "$*$", of concatenation, is associative on the set of path homotopy classes of loops in X based at x_0. [6]
   b) True/False: If two spaces are homeomorphic then they have isomorphic fundamental groups. [4]
3. a) Show that a covering map is an open map. [7]
   b) True/False: If $p : E \to B$ is a covering map and B is compact then so is E. [3]
4. a) Show that the quotient map $q : S^2 \to \mathbb{RP}^2$ is an open map and hence show it is a covering map. [7]
   b) Pick out the true statement(s) for a covering map $p : E \to B$ [3]
      i. If E is contractible then so is B.
      ii. If B is connected then so is E.
      iii. If E is simply connected then so is B.
      iv. if B is simply connected then so is E.
5. a) Show that the fundamental group of the circle is isomorphic to the additive group of integers. [7]
   b) Which of the following spaces is/are simply connected subsets of $\mathbb{R}^2$? [3]

i. $\bigcup_{n=1}^{5}\{(x,y)\in\mathbb{R}^2 : y = nx\}$.

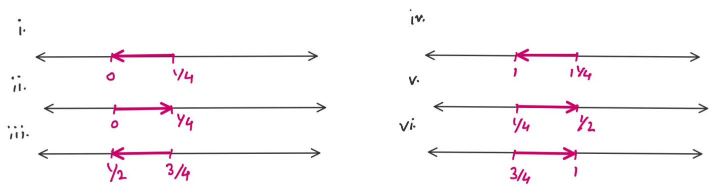
ii. $\{(x,y)\in\mathbb{R}^2 : xy = 0\}$.

iii. $\bigcup_{n=1}^{5}\{(x,y)\in\mathbb{R}^2 : y = x^n\}$.

iv. $\{(x,y)\in\mathbb{R}^2 : xy = 1\}$.

6. a) State the homotopy lifting lemma. Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1. Let $\tilde{f}$ and $\tilde{g}$ be their respective lifts beginning at e_0. Show that if f and g are path homotopic then $\tilde{f}$ and $\tilde{g}$ end at the same point and are also path homotopic. [2+4]

   b) Let $p : \mathbb{R} \to S^1$ be the map $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Which of the following is/are a lift of the path $f(s) = \left(\cos\left(\frac{\pi}{2} + \pi s\right), \sin\left(\frac{\pi}{2} + \pi s\right)\right)$ via p? [4]



7. a) Let $h : S^1 \to X$ be a continuous function. Show that if h is nulhomotopic then h extends to a continuous map $k : B^2 \to X$. [7]

   b) Which of the following are nulhomotopic? [3]

i. The antipodal map $h : S^1 \to S^1$ given by $h(x_1, x_2) = -(x_1, x_2)$.

ii. The identity map $id : S^1 \to S^1$.

iii. The projection map $h : S^1 \to [-1, 1]$ given by $h(x_1, x_2) = x_1$.

iv. The inclusion map $j : S^1 \to \mathbb{R}^2$.

8. a) Give an example of a contractible space. Show that a space is contractible if and only if it has the homotopy type of a point. [1+5]

   b) Classify the following symbols according to homotopy type of a point, a circle and a figure eight: [4]

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9. a) Suppose $X = U \cup V$, where U and V are open subsets of X. Suppose that $U \cap V$ is path connected and $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the respective incusion maps. Show that the images of the induced homomorphisms $i_\# : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_\# : \pi_1(V, x_0) \to \pi_1(X, x_0)$, generate $\pi_1(X, x_0)$. [8]

   b) What will $\pi_1(X)$ be if both $i_\#$ and $j_\#$ are trivial in the above question? [2]

10. a) Show that any continuous function $f : B^2 \to B^2$ has a fixed point. Further, if A is a retract of B^2 then show that any continuous function $f : A \to A$ has a fixed point. [7]

   b) Let $f : [-1, 1] \to [-1, 1]$ be a function satisfying $|f(x) - f(y)| \le \frac{1}{2}|x - y|$. Then the number of fixed points of f is: [3]

   i. 0 ii. 1 iii. infinite iv. finite