Register Number:

Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BANGALORE - 27
M.Sc MATHEMATICS - I SEMESTER
SEMESTER EXAMINATION: OCTOBER 2021
(Examination conducted in January-March 2022)
MT 7321: LINEAR ALGEBRA

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains three printed pages.
2. Attempt any **SEVEN FULL** questions.
3. In objective type questions, one or more options could be correct. Full marks will be awarded only if all the options are correctly marked.

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1. a) Let K be a field and $V = M_n(K)$. Let $W = \{A \in M_n(K) \mid A = A^t\}$. Show that W is a subspace of V. Also, find a basis of W and compute the $\dim(W)$. **[8m]**

   b) Let $V = \mathcal{P}_n(\mathbb{R})$, $W_1 = \{p(x) \in V \mid p(0) = 0\}$ and $W_2 = \{p(x) \in V \mid p(1) = 0\}$. Pick the correct statement(s) from the options given below.

   (i) $\dim(W_1) = \dim(W_2) = n - 1$.
   (ii) $\dim(W_1 \cap W_2) = n - 2$.
   (iii) $\dim(W_1 \cap W_2) = n - 1$.
   (iv) $\dim(W_1) = \dim(W_2) = n$. **[2m]**

2. a) Let V and W be two vector spaces over $\mathbb{Q}$ and $T : V \to W$ be a function. Prove that T is additive (i.e., $T(v_1 + v_2) = T(v_1) + T(v_2)$, for every $v_1, v_2 \in V$) if and only if T is linear. **[7m]**

   b) Let $T : V \to W$ be a linear transformation and $\mathcal{B}$ be a basis of V. Then pick the correct statement(s) from the options given below.

   (i) If T is onto, then $T(\mathcal{B})$ is a basis for W.
   (ii) If T is one-one, then $T(\mathcal{B})$ is a basis for $R(T)$, the range space of T.
   (iii) If T is onto, then $T(\mathcal{B})$ contains a basis of W.
   (iv) If T is one-one, then $T(\mathcal{B})$ is a basis for W. **[3m]**

3. a) Let V be a finite dimensional vector space over a field K. Prove that eigenvectors corresponding to distinct eigenvalues are linearly independent. Further, if $\dim(V) = n$ and T has n distinct eigenvalues, prove that there exists a basis of V consisting of eigenvectors of T. **[7m]**

   b) Pick the correct statement(s) from the options given below.

   (i) An idempotent linear map on a finite dimensional space is always diagonalizable.
   (ii) An idempotent linear map on a finite dimensional space need not be diagonalizable always.
   (iii) An upper triangular matrix is always diagonalizable.
   (iv) An upper triangular matrix with distinct diagonal entries is always diagonalizable. **[3m]**

4. a) Compute the characteristic polynomial and the minimal polynomial of the matrix **[8m]**

$$A = \begin{pmatrix} -1 & 2 & 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{pmatrix}.$$

b) Let K be a field and $A \in M_n(K)$. Let I, O be the identity matrix and the zero matrix respectively, in $M_n(K)$. Pick the correct statement(s) from the options given below.

(i) If A is similar to I, then $A = I$.

(ii) If A is similar to I, then A need not be equal to I.

(iii) If A is similar to O, then $A = O$.

(iv) If A is similar to O, then A need not be equal to O. **[2m]**

5. a) Let V be a finite dimensional vector space over a field K and $T \in \text{End}(V)$. Let λ be an eigenvalue of T. Prove that the geometric multiplicity of λ is less than or equal to the algebraic multiplicity of λ. **[5m]**

b) Suppose $V(F)$ is a 6 dimensional vector space and $T \in End(V)$. Write the Jordan canonical form of T if the minimal polynomial of T is $(x-2)^3(x-5)$, the algebraic multiplicity of 2 is 5 and the geometric multiplicity of 2 is 2. **[2m]**

c) Let V be a finite dimensional vector space over the field $\mathbb{C}$ and $T \in \text{End}(V)$. Pick the correct statement(s) from the options given below.

(i) If algebraic multiplicity of each eigenvalue of T is equal to the corresponding geometric multiplicity, then there exists a basis of V consisting of eigenvectors of T.

(ii) Algebraic multiplicity of each eigenvalue of T is equal to the corresponding geometric multiplicity if and only if there exists a basis of V consisting of eigenvectors of T.

(iii) If T is diagonalizable, then algebraic multiplicity of each eigenvalue of T is equal to the corresponding geometric multiplicity.

(iv) T can be diagonalizable even if the algebraic multiplicity of an eigenvalue is strictly bigger than the corresponding geometric multiplicity. **[3m]**

6. a) Define an inner product space. Prove that the function $\langle\ ,\ \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle u, v\rangle = \alpha_1\bar{\beta}_1 + \cdots + \alpha_n\bar{\beta}_n,$$

where $u = (\alpha_1, \cdots, \alpha_n),\ v = (\beta_1, \cdots, \beta_n) \in \mathbb{C}^n$, is an inner product on $\mathbb{C}^n$. **[2+6m]**

b) Let V be an inner product space over a field K. Prove that $\langle u, \alpha v + \beta w\rangle = \bar{\alpha}\langle u, v\rangle + \bar{\beta}\langle u, w\rangle$, $\forall\ u, v, w \in V$ and $\forall\ \alpha, \beta \in K$. **[2m]**

7. Let V be the inner product space of real valued continuous functions on the interval $[-\pi, \pi]$ with the inner product defined by,

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\ dt.$$

Show that the set $\{1, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots\}$ is an orthogonal set in V. Also find the corresponding orthonormal set. **[10m]**

8. a) Let V be a finite dimensional inner product space and $T, S \in \text{End}(V)$. Show that

i) $(T + S)^* = T^* + S^*$.

ii) $(TS)^* = S^*T^*$. **[6m]**

b) Let T be a linear map on a finite dimensional inner product space and W be an invariant subspace of T. Prove that $W^\perp$ is invariant under T^*, where T^* is the adjoint of T. **[2m]**

c) Pick the correct statement(s) from the options given below.

(i) If $A \in M_n(\mathbb{R})$ is a symmetric matrix, then the eigenvalues of A are always real.

(ii) If $A \in M_n(\mathbb{C})$ is a symmetric matrix, then the eigenvalues of A are always real.

(iii) Hermitian matrices are always diagonalizable.

(iv) If A is a Hermitian matrix, then eigenvectors corresponding to distinct eigenvalues of A are orthogonal. **[2m]**

9. a) Let P be a self-adjoint operator on a finite dimensional inner product space V. Show that P is positive definite if and only if all eigenvalues of P are positive. Hence, deduce that the matrix **[8m]**

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \text{ is positive definite.}$$

b) Let U be an orthogonal operator on $\mathbb{R}^3$. Pick the correct statement(s) from the options given below.

(i) If $v = (1,1,1)$, then the length of the vector Uv is $\sqrt{3}$.

(ii) If $v = (1,1,1)$, then the length of the vector Uv is $\frac{1}{\sqrt{3}}$.

(iii) If $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{R}^3$, then so is $\{Uv_1, Uv_2, Uv_3\}$.

(iv) 1 or -1 has to be an eigenvalue of U. **[2m]**

10. a) If $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$, find the singular values of A. If $A = U\Sigma V^t$ is a singular value decomposition of A, then write the matrix Σ corresponding to this A. **[3m]**

b) Let V be a finite dimensional vector space over a field K. Define a bilinear form on V. If $V = K^n$ and $A \in M_n(K)$, show that the function $f : K^n \times K^n \to K$ defined by $f(u,v) = u^t A v$, for every $u, v \in K^n$ (where the elements of K^n are seen as column matrices of order $n \times 1$), is a bilinear form on K^n. **[5m]**

c) Pick the correct statement(s) from the options given below.

(i) If U is an orthogonal matrix, then all singular values U are equal to each other.

(ii) If U is an orthogonal matrix, then any pair of distinct rows of U is linearly independent.

(iii) If U is orthogonal and symmetric, then U has to be $\pm I$, where I is the $n \times n$ identity matrix.

(iv) If U is orthogonal and positive definite, then U has to be I. **[2m]**