Register Number:

Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.SC MATHEMATICS - II SEMESTER
SEMESTER EXAMINATION: APRIL, 2022
(Examination conducted in July 2022)

MT 8118: ALGEBRA II

Duration: 2.5 Hours **Max. Marks:** 70

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1. The paper contains two printed pages and one part.
2. Answer any **SEVEN FULL** questions.
3. All multiple choice questions may have one or more correct options. Full marks will be awarded only for writing **all correct options** in your answer script.

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1. a) Show that any simple abelian group is isomorphic to $\mathbb{Z}_p$, where p is a prime number. [**6 marks**]

   b) An abelian group G of order 45 always has an element of order [**4 marks**]

   (I) 15 (II) 3 (III) 9 (IV) 45

2. a) Define derived series of a group G. Prove that G is solvable if and only if $G^{(n)} = 1$ for some $n \geq 0$. [**6 marks**]

   b) Which of the following is/are true? [**4 marks**]

   (I) Every simple group is solvable.
   (II) Every nilpotent group is solvable.
   (III) Every cyclic group is solvable.
   (IV) Every solvable group is simple.

3. a) Prove (without using Feit-Thompson Theorem) that the following are equivalent:
   1. Every group of odd order is solvable.
   2. The only simple groups of odd order are those of prime order. [**6 marks**]

   b) Let G be a group and N be a normal subgroup of G. Show that if N and G/N is solvable then G is also solvable. [**4 marks**]

4. a) Compute the number of cyclic subgroups of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$. [**5 marks**]

   b) Let G be the group $(\{1, 7, 17, 23, 49, 55, 65, 71\}, \otimes_{96})$. Find an explicit description of G as cartesian product of cyclic groups. [**5 marks**]

5. Let P be an R-module. Show that the following are equivalent: [**10 marks**]

   (I) P is a projective R-module.
   (II) If P is the quotient of an R-module M then, P is isomorphic to a direct summand of M.
   (III) P is a direct summand of a free R-module.

6. a) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Prove that if M' and M'' are finitely generated then M is also finitely generated. **[6 marks]**

   b) Which of the following is/are true? **[4 marks]**

   (I) $M_n(\mathbb{R})$ is a finitely generated $\mathbb{R}$-module but not free.

   (II) $M_n(\mathbb{R})$ is a free and finitely generated $\mathbb{R}$-module.

   (III) $M_n(\mathbb{R})$ is an infinitely generated $\mathbb{Q}$-module but not free.

   (IV) $M_n(\mathbb{R})$ is a free and infinitely generated-$\mathbb{Q}$ module.

7. Find the splitting field of $x^p - 2$ over $\mathbb{Q}$. Also, find the basis and the dimension of splitting field over $\mathbb{Q}$? **[10 marks]**

8. a) Show that given a prime number p and natural number n, there exists a finite field with p^n elements. Further show that any finite field with p^n elements is unique up to isomorphism. **[6 marks]**

   b) Pick out the correct statement(s) from the following: **[4 marks]**

   I. Every finite extension is separable.

   II. Every finite extension of a positive characteristic field is separable.

   III. Every finite extension of $\mathbb{Q}$ is separable.

   IV. Let $\text{char}(F) = 5$. Then any degree 3 extension K/F is separable.

9. a) State the Fundamental Theorem of Galois Theory. **[3 marks]**

   b) Draw the complete lattice diagram of all the intermediate subfields of $\mathbb{F}_{2^{12}}/\mathbb{F}_2$. Also, mention the degrees of the extensions at each stage. **[3 marks]**

   c) Determine which of the following field extension K/F is/are Galois. **[4 marks]**

   I. Let, $K = \mathbb{Q}(\zeta_n)$ and $F = \mathbb{Q}$, where ζ_n is a primitive n^{th} root of unity.

   II. Let α be a real 10^{th} root of 3, $K = F(\alpha)$ and $F = \mathbb{Q}$.

   III. Let $K = \mathbb{R}(\zeta_n)$ and $F = \mathbb{R}$, where ζ_n is a primitive n^{th} root of unity.

   IV. Let $F = \mathbb{F}_3(t)$ and K be the splitting field of $x^3 - t$ over F.

10. a) Let ζ_7 be a primitive 7^{th} root of unity. Give an explicit description of the Galois group $\text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$. Find an intermediate subfield F of $\mathbb{Q}(\zeta_7)/\mathbb{Q}$ such that $[F : \mathbb{Q}] = 3$. **[7 marks]**

    b) Let ζ_8 denote a primitive 8^{th} root of unity. Pick out the correct statement(s) from the following: **[3 marks]**

    I. The dimension of $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ is 4.

    II. The Galois group $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

    III. The minimal polynomial of ζ_{12} over $\mathbb{Q}$ is $x^4 + 1$.

    IV. The number of intermediate sub-fields of $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$ (including $\mathbb{Q}(\zeta_{12})$ and $\mathbb{Q}$) is 3.

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