Date:
Registration number:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
MSC MATHEMATICS - II SEMESTER
END SEMESTER EXAMINATION: APRIL 2022
(Examination conducted in July 2022)

MT8121 – ALGEBRA II

Time - 2 ½ hrs   Max Marks - 70

This question paper contains 2 printed pages

Answer any 7 complete questions ($7 \times 10 = 70$)

1.a. Define the characteristic of a field. Prove that the characteristic of a field F is either zero or a prime number.
(5)

1.b. Show that $f(x) = x^3 + 9x + 6$ is irreducible over $\mathbb{Q}$. If θ denotes a root of $f(x)$ in an extension of $\mathbb{Q}$, then compute the inverse of $1 + \theta$ in that extension.
(5)

2.a. Show that if the field K is algebraic over F and L is algebraic over K, then L is algebraic over F.
(5)

2.b. Define the degree of an algebraic element. Compute the degree of extension $\left[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}(\sqrt{2})\right]$ and also, find the minimal polynomial for $\sqrt[6]{2}$ over $\mathbb{Q}(\sqrt{2})$.
(5)

3.a. Find the splitting field for the given polynomials: $f(x) = x^4 + 4 \in \mathbb{Q}[x]$, $g(x) = x^2 - 2 \in \mathbb{Q}[x]$.
(5)

3.b. Define an algebraically closed field. Show that an algebraic closure of a field is algebraically closed.
(5)

4.a. State and prove the criterion for a polynomial to be separable over a field F.
(6)

4.b Define the n^{th} cyclotomic polynomial $\phi_n(x)$ and compute $\phi_1(x), \phi_2(x)$ and $\phi_3(x)$.
(4)

5.a Give an example for both separable and inseparable polynomial.
(2)

5.b Show that the cyclotomic polynomial $\phi_n(x)$ is an irreducible monic polynomial in $\mathbb{Z}[x]$ of degree $\varphi(n)$, where φ denotes Euler's phi function.

(8)

6.a Is the field extension $\mathbb{Q}(\sqrt{2})$ Galois over $\mathbb{Q}$? Justify your answer.

(5)

6.b Show that the field extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois over $\mathbb{Q}$.

(5)

7. If the extension K/F is Galois, then show that K is the splitting field of some separable polynomial over F. What about the converse? (Just mention whether the converse is true or not).

(10)

8. State the fundamental theorem of Galois theory. Draw the diagram showing the 1-1 correspondence between the subgroups of Galois group $Gal(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$ and the corresponding fixed subfields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(10)

9.a Show that the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_n)$ of n^{th} roots of unity is isomorphic to the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^{\times}$.

(6)

9.b Find the discriminant of the polynomial $f(x) = x^2 - 3x + 2$.

(4)

10. Prove that the polynomial $f(x)$ can be solved by radicals if and only if its Galois group is a solvable group.

(10)