Register Number:

Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.SC MATHEMATICS - II SEMESTER
SEMESTER EXAMINATION: APRIL, 2022
(Examination conducted in July 2022)

MT 8221: MEASURE AND INTEGRATION

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains two printed pages and one part.
2. Answer any **SEVEN FULL** questions.
3. All multiple choice questions have 1 or more than one correct option. Full marks will be awarded only for writing **all correct options** in your answer script.

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1. a) Prove that any open subset of $\mathbb{R}$ is Lebesgue measurable. **[7]**

   b) Which of the following measures is/are σ-finite on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$? **[3]**

   i. $\mu(A) = |A \cap \mathbb{Q}^c|$ ii. $\mu(A) = |A \cap \mathbb{Q}|$ iii. $\mu(A) = |A \cap \mathbb{N}^c|$ iv. $\mu(A) = |A \cap \mathbb{Q}|$

2. a) Let E_1 and $E_2 \in \mathcal{L}(\mathbb{R}^n)$. Then show that $E_1 \cup E_2 \in \mathcal{L}(\mathbb{R}^n)$. Further, if $E_1 \cup E_2 = \emptyset$ then show that $\mu_*(E_1 \cup E_2) = \mu_*(E_1) + \mu_*(E_2)$. **[7]**

   b) Which of the following sets has measure zero in $(\mathbb{R}^2, \mathcal{P}(\mathbb{R}^2), \delta_{(0,0)})$? **[3]**

   i. $\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 : y = nx\}$

   ii. $\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 : y = nx + 1\}$

   iii. $\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 : y = n(x + 1)\}$

   iv. $\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 : y = (n + 1)x\}$

3. a) Let $(X, \mathcal{S}, \mu)$ be a measure space. Show that if $\{E_i\}$ is a countable collection of subsets of X in $\mathcal{S}$ such that $E_1 \subseteq E_2 \subseteq E_3 \cdots$ then, $\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{n\to\infty} \mu(E_n)$. **[4]**

   b) Let $(X, \mathcal{S}, \mu)$ be a measure space. Show that a function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}(-\infty, a] \in \mathcal{S}$ for all $a \in \mathbb{R}$. **[4]**

   c) Let $A, B \subseteq \mathbb{R}$.. Which of the following is/are true? **[2]**

   i. $\chi_{A\cap B} = \min\{\chi_A, \chi_B\}$

   ii. $\chi_{A\cup B} = \max\{\chi_A, \chi_B\}$

   iii. $\chi_{AB} = \chi_A \chi_B$

   iv. $\chi_{A\setminus B} = \chi_A - \chi_B$

   where $AB := \{a \cdot b : a \in A \text{ and } b \in B\}$.

4. a) Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, \mathcal{S}, \mu)$. Prove that $\sup f_n$, $\inf f_n$, $\limsup f_n$ and $\liminf f_n$ are also measurable. **[7]**

   b) Let $(X, \mathcal{S}, \mu)$ be a measure space and f, g be strictly positive functions on X. Which of the following is/are true? **[3]**

i. fg measurable $\implies f$ and g measurable
ii. f and g measurable $\implies fg$ measurable
iii. f/g measurable $\implies f$ and g are measurable
iv. f and g measurable $\implies f/g$ measurable.

5. State and Prove Egorov's Theorem. **[10]**

6. a) Prove the Bounded Convergence Theorem: "Suppose $\{f_n\}$ is a sequence of measurable functions that are all bounded by M and supported on a set E of finite measure and $f_n \to f$ a.e. Then, f is a.e bounded, a.e supported on E and $\lim_{n\to\infty} \int f_n = \int f$." **[7]**

   b) Which of the following are integrable? **[3]**

   i. $1/x$ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \delta_0)$
   ii. $1/x$ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \delta_1)$
   iii. $\chi_{\mathbb{Q}}$ on $(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$
   iv. $\chi_{\mathbb{Q}^c}$ on $(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$

7. a) Let f, g be non-negative integrable functions defined on a measure space X. Show that **[7]**

$$\int_X (af + bg) = a \int_X f + b \int_X g \text{ for any } a, b \geq 0.$$

   b) Let $X = Y = [0,1]$. Give X the Lebesgue measure m and Y the counting measure μ. Let $f(x,y) = 1$ if $x = y$ and 0 otherwise. Which of the following is/are true? **[3]**

   i. $\int_X f(x,y)dm = 0$ for all $y \in Y$
   ii. $\int_Y f(x,y)d\mu = 0$ for all $x \in X$
   iii. $\int_X \int_Y f(x,y)d\mu dm = \int_Y \int_X f(x,y)dm d\mu$
   iv. $\int_Y \int_X f(x,y)dm = 0$.

8. a) Let s_1 and s_2 be two simple functions defined on a measure space $(X, \mathcal{S}, \mu)$. Show that if $s_1 \leq s_2$ then $\int_X s_1 \leq \int_X s_2$. **[3]**

   b) Let $(X, \mathcal{S}, \mu)$ be a measure space and $h \in \mathcal{L}^1(X)$ be a non-negative function. For each $E \in \mathcal{S}$ define $\nu(E) = \int_E h$. Show that ν is a measure. **[4]**

   c) Given two measures ν_1 and ν_2 on the same measure space we say that $\nu_1 << \nu_2$ if $\nu_2(E) = 0 \implies \nu_1(E) = 0$. In which of the following cases is $\nu_1 << \nu_2$ on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$? **[3]**

   i. $\nu_1 =$ Lebesgue measure and $\nu_2 = \int_E h d\nu_1$ for some $h \in L^1$
   ii. $\nu_2 =$ Lebesgue measure and $\nu_1 = \int_E h d\nu_2$ for some $h \in L^1$
   iii. $\nu_1 =$ Lebesgue measure and $\nu_2 =$ counting measure.
   iv. $\nu_1 =$ counting measure and $\nu_2 =$ Lebesgue measure.

9. a) State and prove Hölder's inequality. **[7]**

   b) Which of the following is(are) true? **[3]**

   i. $L^1([0,1]) \subseteq L^2([0,1])$ with Lebesgue measure
   ii. $L^2([0,1]) \subseteq L^1([0,1])$ with Lebesgue measure
   iii. $L^2([0,1]) \subseteq L^1([0,1])$ with Dirac measure at 0
   iv. $L^1([0,1]) \subseteq L^2([0,1])$ with Dirac measure at 0.

10. a) Define function of bounded variation. Show that a function $f : [a,b] \to \mathbb{R}$ is of bounded variation if and only if f is the difference of two monotonic functions. **[2+5]**

    b) Show that a Lipschitz continuous function is absolutely continuous. **[3]**