Register Number:
Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BANGALORE - 27
M.Sc MATHEMATICS - IV SEMESTER
SEMESTER EXAMINATION: APRIL 2022
(Examination conducted in July 2022)
MTDE 01018: DIFFERENTIAL GEOMETRY

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains two printed pages.
2. Attempt any **SEVEN FULL** questions.
3. In objective type questions, one or more options could be correct. Full marks will be awarded only if all the options are correctly marked.

---

1. a) Let f and g be differentiable real-valued functions on $\mathbb{E}^3$, $\mathbf{v_p}$ a tangent vector and $\mathbf{V}$ a vector field. Then prove that
$$\mathbf{v_p}[fg] = \mathbf{v_p}[f].g(\mathbf{p}) + f(\mathbf{p}).\mathbf{v_p}[g].$$
Further, deduce that
$$\mathbf{V}[fg] = \mathbf{V}[f]g + f\mathbf{V}[g].$$
[7m]

b) Let ϕ be a 1-form on $\mathbb{E}^3$, V a vector field and f a differentiable real-valued function. Then pick the correct statement(s) from the options given below.

(i) $f\phi(V)$ is a real-valued function on $\mathbb{E}^3$.
(ii) $\phi(V)$ is a real-valued function on $\mathbb{E}^3$.
(iii) $\phi(f)$ is a real-valued function on $\mathbb{E}^3$.
(iv) $V[f] = df(V)$ as functions on $\mathbb{E}^3$. **[3m]**

2. a) Let $\alpha(t) = (2t, t^2, \log t)$ be a curve in $\mathbb{E}^3$ defined on $(0, \infty)$. Find the arc length of $\alpha(t)$ between the points $(2, 1, 0)$ and $(4, 4, \log 2)$. **[5m]**

b) Let β be a unit-speed curve in $\mathbb{E}^3$ with curvature $\kappa > 0$. Then prove that β is a plane curve if and only if its torsion $\tau = 0$. **[5m]**

3. Let β be a unit-speed curve with $\kappa > 0$ and $\tau \neq 0$. If β lies on a sphere with center $\mathbf{c}$ and radius r, prove that $\beta - \mathbf{c} = -\rho N - \rho'\sigma B$, where $\rho = \frac{1}{\kappa}$ and $\sigma = \frac{1}{\tau}$. Further, deduce the expression for the radius of the sphere in terms of κ and τ. **[10m]**

4. Let $\alpha(t) = (\cosh t, \sinh t, t)$, where $t \in \mathbb{R}$. Show that $\alpha(t)$ is a cylindrical helix. **[10m]**

5. Given the frame $\mathbf{e_1} = \frac{1}{3}(2, 2, 1), \mathbf{e_2} = \frac{1}{3}(-2, 1, 2), \mathbf{e_3} = \frac{1}{3}(1, -2, 2)$ at $\mathbf{p} = (0, 1, 0)$ and the frame $\mathbf{f_1} = \frac{1}{\sqrt{2}}(1, 0, 1), \mathbf{f_2} = (0, 1, 0), \mathbf{f_3} = \frac{1}{\sqrt{2}}(1, 0, -1)$ at $\mathbf{q} = (3, -1, 1)$, find the isometry $F = TC$ which carries frame $\mathbf{e}$ to the frame $\mathbf{f}$. **[10m]**

6. a) Show that the surface of revolution $M : (\sqrt{x^2 + y^2} - 4)^2 + z^2 = 4$ is a torus. Also, write a parametrization for this surface. **[6m]**

   b) Let $\mathbf{x}\colon \mathbf{D} \to \mathbb{E}^3$ be a proper patch on an open subset $\mathbf{D}$ of $\mathbb{E}^2$. Pick the correct statement(s) from the options given below.

   (i) $\mathbf{x}$ is a homeomorphism from $\mathbf{D}$ to $\mathbf{x}(\mathbf{D})$.
   (ii) The Jacobian of $\mathbf{x}$ need not have rank 2 always.
   (iii) $\mathbf{x}(\mathbf{D})$ is an example of a simple surface.
   (iv) The Jacobian of $\mathbf{x}$ always has rank 2. **[4m]**

7. In which of the following cases is the morphism $\mathbf{x} : \mathbb{E}^2 \to \mathbb{E}^3$ a patch? Justify your answers in each case.

   (i) $\mathbf{x}(u, v) = (u, uv, v)$.
   (ii) $\mathbf{x}(u, v) = (u^2, u^3, v)$.
   (iii) $\mathbf{x}(u, v) = (\cos 2\pi u, \sin 2\pi u, v)$.
   (iv) $\mathbf{x}(u, v) = (u, u^2, v + v^3)$. **[10m]**

8. a) Let M be a surface in $\mathbb{E}^3$. Prove that the shape operator at each point $\mathbf{p} \in M$ is a linear operator on the tangent space $T_{\mathbf{p}}(M)$. **[6m]**

   b) Pick the correct statement(s) from the options given below.

   (i) If M is a sphere of radius r centered at the origin in $\mathbb{E}^3$, then the shape operator at each point $\mathbf{p} \in M$ is an invertible linear operator on $T_{\mathbf{p}}(M)$.
   (ii) If M is a plane in $\mathbb{E}^3$, then the shape operator at each point $\mathbf{p} \in M$ is an invertible linear operator on $T_{\mathbf{p}}(M)$.
   (iii) If M is the cylinder $x^2 + y^2 = 1$, then the shape operator at each point $\mathbf{p} \in M$ is an invertible linear operator on $T_{\mathbf{p}}(M)$.
   (iv) If M is the saddle surface $z = xy$ and $\mathbf{p} = (0, 0, 0)$, then the shape operator at the point $\mathbf{p} \in M$ is an invertible linear operator on $T_{\mathbf{p}}(M)$. **[4m]**

9. Let $M \subset \mathbb{E}^3$ be a surface and $\mathbf{p} \in M$. Define the principal curvatures and principal directions of M at $\mathbf{p}$. If S is the shape operator of M, prove that the principal curvatures of M are precisely the eigenvalues of S and the principal directions are the corresponding eigenvectors. **[10m]**

10. a) Define mean curvature H of a surface $M \subset \mathbb{E}^3$. When do we say a surface is minimal? **[2m]**

    b) Show that the helicoid $\mathbf{x}(u, v) = (u\ \cos\ v,\ u\ \sin\ v,\ bv)$ is a minimal surface, where $b \neq 0$. **[8m]**