Register Number:

Date:

ST. JOSEPH'S UNIVERSITY, BENGALURU-27
M.SC (MATHEMATICS) - I SEMESTER
SEMESTER EXAMINATION: OCTOBER, 2022
(Examination conducted in December 2022)
MT 7121: ALGEBRA I

Duration: 2 Hours **Max. marks:** 50

1. The paper contains **TWO** printed pages and ONE part.
2. Attempt any **FIVE FULL** questions.
3. All multiple choice questions have **one or more** correct option. Write **all** the correct options answer booklet.
4. Calculators are allowed.

1. a) Write down a representative element of each conjugacy classes of S_5 corresponding to partitions of 5. Also, write down the number of elements in each of the conjugacy classes. [6 m]

   b) Which of the following are true statements? [4 m]

   I) The order of a k-cycle in S_n is k
   II) Any permutation in S_n can be written as a product of disjoint cycles
   III) If $\sigma \in S_5$ has order 2 then σ is a 2-cycle
   IV) If $\sigma \in S_4$ is a 4-cycle then σ^2 is also a 4-cycle.

2. a) Show that a group of order p^2 is abelian, where p is some prime. Moreover, show that any such group is either isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. [6 m]

   b) Show that if the center of a group G is index n, then every conjugacy class has at most n elements. [3 m]

   c) Choose all the correct statements. [2 m]

   I) Every group of order 51 is cyclic
   II) Every group of order 151 is cyclic
   III) Every group of order 505 is cyclic.

3. a) Show that the set $\mathrm{Aut}(\mathbb{Z}_n)$ is isomorphic to the group $U(n)$ of units of $\mathbb{Z}_n$. [4 m]

   b) Let G be a group of order 3825. Prove that if H is a normal subgroup of order 17 in G then $H \leq Z(G)$. [4 m]

   c) Choose all the correct statements. [2 m]

   I) $\mathrm{Aut}\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right) = S_3$
   II) $[\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)] = 2$
   III) $|\,\mathrm{Aut}\left(\mathbb{Z}_{12}\right)| = 11$
   IV) $|\,\mathrm{Aut}\left(\mathbb{Z}_{12}\right)| = 4$.

4. a) Show that if a group of order 60 has more than one Sylow 5-subgroup, then it is simple. [7 m]

   b) Consider the group $G = GL_2(\mathbb{Z}p)$, where p is a prime number. Choose all the correct statements.

I) The order of G is $p(p-1)^2(p+1)$

II) Every element of order p is conjugate to a matrix $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, where $a(\neq 0) \in \mathbb{Z}_p$

III) G has $p+1$ Sylow p-subgroups

IV) G has exactly one element of order p.

OR

a) Compute the number of cyclic subgroups of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$. [5 m]

b) Let G be the group $\left(\{1, 7, 17, 23, 49, 55, 65, 71\}, \otimes_{96}\right)$. Find an explicit description of G as cartesian product of cyclic groups. [5 m]

5. a) Show that a polynomial of degree n over a field has at most n roots. [4 m]

b) Show that the polynomial $\frac{3}{7}x^4 - \frac{2}{7}x^2 + \frac{9}{35}x + \frac{3}{5}$ is irreducible over $\mathbb{Q}$. [3 m]

c) Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree n. Choose all the correct statements. [3 m]

I) $f(x)$ is irreducible in $\mathbb{Z}[x]$ implies $f(x)$ is irreducible in $\mathbb{Q}[x]$

II) $f(x)$ is irreducible in $\mathbb{Q}[x]$ implies $f(x)$ is irreducible in $\mathbb{Z}[x]$

III) If $f(x)$ is reducible over $\mathbb{Z}$, then it has a real root

IV) If $f(x)$ has a real root, then it is reducible over $\mathbb{Z}$.

6. a) Let F be a field. Show that if $p(x) \in F[x]$ is irreducible then $\langle p(x) \rangle$ is a maximal ideal in $F[x]$. [4 m]

b) Construct a field with 25 elements. [3 m]

c) Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree n. Pick out the true statements about the roots of $f(x)$. [3 m]

I) They can belong to $\mathbb{Z}$

II) They always belong to $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Z}$

III) They always belong to $(\mathbb{C} \setminus \mathbb{Q}) \cup \mathbb{Z}$

IV) They can belong to $(\mathbb{Q} \setminus \mathbb{Z})$.

7. a) Show that $\mathbb{Z}[i]$ is an ED. [4 m]

b) Show that every prime ideal in a PID is a maximal ideal. [4 m]

c) Choose all the correct statements. [2 m]

I) The polynomial ring $\mathbb{Z}[x]$ is a PID

II) The polynomial ring $\mathbb{Z}[x]$ is a UFD

III) The polynomial ring $\mathbb{Q}[x]$ is a PID

IV) The polynomial ring $\mathbb{Q}[x]$ is a UFD.