Register Number:
Date:

ST. JOSEPH'S UNIVERSITY, BENGALURU-27
M.Sc. (MATHEMATICS) - I SEMESTER
SEMESTER EXAMINATION: OCTOBER 2022
(Examination conducted in December 2022)
MT7321: LINEAR ALGEBRA

Duration: 2 Hours **Max. Marks:** 50

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1. The paper contains two printed pages.
2. Attempt any **FIVE FULL** questions. Each question carries **TEN** marks.

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1. a) Let W_1 and W_2 be subspaces of a finite dimensional vector space V over a field $\mathbb{F}$ and $W = W_1 \oplus W_2$. Prove that $\dim(W) = \dim(W_1) + \dim(W_2)$. **[8m]**
   b) Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0-6) = (2,1)$? **[2m]**

2. a) Prove or disprove the following statement:
   The union of two subspaces of a vector space V is a subspace of V. **[3m]**
   b) Let V and W be vector spaces over a field $\mathbb{F}$ and $\{v_1, \ldots, v_n\}$ be a basis for V. Suppose $w_1, w_2, \ldots, w_n$ are given vectors in W. Prove that there exists a linear transformation $T : V \to W$ such that $T(v_i) = w_i,\ 1 \leq i \leq n.$ **[5m]**
   c) Let $T : V \to V$ be a linear operator on a vector space V over a field $\mathbb{F}$. Show that the null space of T is T-invariant. **[2m]**

3. a) Let T be a linear operator on a finite dimensional vector space V over a field $\mathbb{F}$. Show that c is an eigenvalue of T if and only if $T - cI$ is singular. **[3m]**
   b) For a linear operator T on a vector space V, show that an eigenvalue associated with an eigenvector is unique but an eigenvector associated with an eigenvalue is not unique. **[4m]**
   c) Find the algebraic multiplicity of the eigenvalues for the following matrix:
   $$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$
   [3m]

4. a) Diagonalize the following matrix:
   $$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$
   [7m]

b) For the above matrix A, compute A^{21} using its diagonal form. **[3m]**

5. a) State and prove the Cauchy-Schwarz inequality for the vectors in an inner product space. **[5m]**

   b) Let V be an inner product space, and let T be a normal operator on V. Then prove the following:

      i) $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$.

      ii) If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2, then x_1 and x_2 are orthogonal.

   [2m+3m]

6. a) Use the Gram-Schmidt procedure to convert the following basis vectors of $\mathbb{R}^3$ into an orthonormal basis vectors:

$$x = (1, 0, 1), y = (1, 1, 1) \text{ and } z = (0, 1, 2).$$

   [7m]

   b) Find the singular values of the following matrix:

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

   [3m]

7. a) Define a symmetric bilinear form on a vector space V over a field $\mathbb{F}$. **[3m]**

   b) Is the standard inner product on $\mathbb{R}^n$ a symmetric bilinear form? **[1m]**

   c) Prove that the bilinear form $f(x, y) = x^T Ay$ is a symmetric bilinear form on $\mathbb{F}^n$ if and only if the matrix A is symmetric. **[6m]**