Register Number:

Date & Session:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27

B.Sc. Mathematics - VI SEMESTER

SEMESTER EXAMINATION: APRIL 2023

(Examination conducted in May 2023)

MT 6118 – MATHEMATICS VII

(For current batch students only)

Time: $2\frac{1}{2}$ hrs **Max Marks: 70**

This paper contains TWO printed pages and THREE parts.

Part-A

Answer any FIVE of the following. **5 X 2 = 10**

1. Is it true that the set of all rational numbers $\mathbb{Q}$ is a vector space over the field of real numbers $\mathbb{R}$ under the usual addition and multiplication? Justify your answer.
2. Check whether $x+1 \in span(S)$, where $S=\{x^2+2x,\, x^2-1\}$ is a subset of the vector space $P_2(\mathbb{R})$.
3. Give the definitions of a basis and the dimension for a vector space $V(F)$.
4. Show that the mapping $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1,a_2,a_3)=(a_1-a_3, 3a_2)$ is a linear transformation from the vector space $\mathbb{R}^3$ to the vector space $\mathbb{R}^2$.
5. Let B_1 and B_2 be the standard bases for the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Find the matrix $[T]_{B_1}^{B_2}$ of the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y)=(x-2y,\, y-x,\, 3x+5y)$.
6. Solve: $\frac{dx}{x^{-1}y^2z}=\frac{dy}{xz}=\frac{dz}{y^2}$
7. Form the partial differential equation by eliminating the arbitrary constants c and α from $x^2+y^2=(z-c)^2\tan^2\alpha$.
8. Find the particular Integral of the partial differential equation $(D^2-2DD'-3D'^2)z=\sin(2x+y)$.

Part-B

Answer any SEVEN of the following. **7 X 6 = 42**

9. State and prove the necessary and sufficient conditions for a non-empty subset W of a vector space V(F) to be a subspace of V.
10. Let S_1 and S_2 be two subsets of a vector space V(F). Prove that $span(S_1)=span(S_2)$ if and only If $S_1 \subseteq span(S_2)$ and $S_2 \subseteq span(S_1)$. If $S_1=\{x^2+x+1, x^2\}$, $S_2=\{x+1, 3x^2\}$ and $V(F)=P_2(\mathbb{R})$, then show that $span(S_1)=span(S_2)$.

11. (i) Let V be a vector space over a field F and S be a linearly independent subset of V. Let $x \in V$ be such that $x \notin S$. Prove that $S \cup \{x\}$ is linearly dependent if and only if $x \in span(S)$.

(ii) Verify whether $S = \{(1,1,0),\ (1,0,1),\ (0,1,1)\}$ is a linearly dependent subset of the vector space $\mathbb{F}^3$, where $\mathbb{F}$ is a field of characteristic 2. **(4+2)**

12. Prove that a subset $B = \{u_1, u_2, ..., u_n\}$ of a vector space V(F) is a basis for V if and only if each vector in V can be uniquely expressed as a linear combination of the vectors of B.

13. (i) Find a basis and the dimension for the subspace W of the vector space $\mathbb{R}^4$, where $W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1 + 2a_4 = 0,\ a_2 = a_3\}$.

(ii) Find the dimension of the subspace spanned by the subset S of the vector space $M_2(\mathbb{R})$, where $S = \left\{ \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -9 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ -4 & 6 \end{pmatrix} \right\}$. **(3+3)**

14. Define a linear transformation. Show that the subset $\{(1,2),(-2,1)\}$ is a basis for $\mathbb{R}^2$. Find a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,2) = (2,7,-8)$ and $T(-2,1) = (1,1,1)$.

15. Let V and W be vector spaces over the same field F and the dimension of V be finite. For any linear transformation $T: V \to W$, prove that $nullity(T) + rank(T) = \dim(V)$.

16. Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation defined by $T\big(f(x)\big) = \frac{d}{dx}\big(f(x)\big) + 6\int_0^x f(t)\,dt$. Find the range space and the rank of T. Show that T is one-to-one using the Rank-Nullity theorem.

17. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $T(a,\ b) = (3a - b,\ a + 3b)$, and let $B_1 = \{(1,1),(1,-1)\}$ and $B_2 = \{(2,4),(3,1)\}$ be the ordered bases for $\mathbb{R}^2$. Then

(i) find the change of coordinates matrix Q that changes B_2 coordinates into B_1 coordinates,

(ii) compute $[T]_{B_1}$, and (iii) find $[T]_{B_2}$ using $[T]_{B_1}$ and Q. **(2+2+2)**

Part-C

Answer any THREE of the following. 3 X 6 = 18

18. Verify the condition for integrability and solve: $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$

19. Form the partial differential equation by eliminating the arbitrary functions f and g in

$$z = \frac{1}{x}[f(x - at) + g(x + at)].$$

20. Find the complete solution of $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

21. Solve by Charpit's method: $(p^2 + q^2)y = qz$

22. Solve the partial differential equation: $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$