Register Number:

Date:

ST. JOSEPH'S UNIVERSITY, BENGALURU-27
M.Sc. (MATHEMATICS) - I SEMESTER
SEMESTER EXAMINATION: OCTOBER 2023
(Examination conducted in November/December 2023)
MT7321: LINEAR ALGEBRA
(For current batch students only)

Duration: 2 Hours **Max. Marks:** 50

1. The paper contains two printed pages.
2. Attempt any **FIVE FULL** questions. Each question carries **TEN** marks.
3. **Question No. 3** has internal choice and answer either **part a or part b**.

---

1. a) Let $T : V \to W$ be linear and let $\{v_1, \cdots, v_k\} \subseteq V$. Show that if $\{T(v_1), \cdots, T(v_k)\}$ is linearly independent in W, then $\{v_1, \cdots, v_k\}$ is linearly independent in V. Also, prove the converse if T is 1-1. **[5m]**

   b) Is the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, y - 1)$ a linear transformation? Justify your answer. **[2m]**

   c) Let $W = \{(x, y) \in \mathbb{R}^2 : y = mx + b\}$, where $m, b \in \mathbb{R}$. Prove that W is a subspace of the vector space $\mathbb{R}^2$ if and only if $b = 0$. **[3m]**

2. a) Let $W_1, \cdots, W_n$ be subspaces of a vectorspace V. Prove that $V = W_1 \oplus \cdots \oplus W_n$ iff each $v \in V$ admits a unique representation $v = v_1 + \cdots + v_n$, where $v_i \in W_i$ for $i = 1, 2, \cdots, n$. **[4m]**

   b) Let $V = W_1 \oplus W_2$ be a vector space and let $T : V \to V$ be a projection on subspace W_1 along the subspace W_2. Then prove the following: **[6m]**

      i) $T^2 = T$.

      ii) $W_1 = N(I - T)$ and $W_2 = R(I - T)$.

3. a) i) Consider the subspace $W = \{A \in M_{4\times4}(\mathbb{R}) : trace(A) = 0\}$. Find the basis and the dimension of W. **[4m]**

      ii) Define a T-invariant subspace. Is the sum of two T-invariant subspaces a T-invariant subspace? Justify your answer. **[3m]**

      iii) Let $A \in M_{2\times2}(\mathbb{R})$ with $trace(A) = 5$ and $\det(A) = 4$. Find the eigenvalues of A. **[3m]**

OR

   b) i) State and prove the Cayley-Hamilton theorem. **[8m]**

ii) Compute the minimal polynomial of the following matrix: **[2m]**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

4. a) Diagonalize the following matrix: **[8m]**

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}.$$

b) Let T be a linear operator on a vector space V of dimension 6. Write the Jordan canonical form of T if the minimal polynomial of T is $(x-2)^4(x-7)^2$. **[2m]**

5. a) Prove that the absolute value of an eigenvalue of a unitary operator T on a finite-dimensional inner product space V is 1. **[4m]**

b) Let V be an inner product space, and let T be a normal operator on V. Then prove the following statements: **[6m]**

i) $T - cI$ is normal for every $c \in \mathbb{C}$.

ii) If $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda} x$.

6. a) Use the Gram-Schmidt procedure to convert the following basis vectors of $\mathbb{R}^3$ into an orthonormal basis vectors: **[7m]**

$$x = (1,1,0), y = (1,1,1) \text{ and } z = (3,1,1).$$

b) Is the following matrix a positive definite? Justify your answer: **[3m]**

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix}.$$

7. a) Define the matrix of a bilinear form on a finite-dimensional vector space $V(\mathbb{F})$. Find the matrix of the bilinear form defined by the standard dot product on $\mathbb{R}^2$ w.r.t the basis $\{(1,1),(0,1)\}$. **[3m]**

b) Consider the vector space $V = M_{2\times 2}(\mathbb{R})$. Show that the function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ defined by $\langle A, B\rangle = trace(AB), \forall\, A, B \in V$ is a symmetric bilinear form. **[4m]**

c) Consider the bilinear form $\langle\ ,\ \rangle$ on $\mathbb{R}^2$ defined by $\langle x, y\rangle = x^T A y$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Is the form a positive definite or a negative definite? Justify your answer. **[3m]**