Register number:

Date and session:

ST. JOSEPH'S UNIVERSITY, BENGALURU-27
M.Sc (MATHEMATICS) - III SEMESTER
SEMESTER EXAMINATION: OCTOBER, 2023
(Examination conducted in November/December 2023)

MTDE 9322: GRAPHS AND MATRICES

(For current batch students only)

Duration: 2 Hours **Max. Marks:** 50

1. The paper contains **TWO** printed pages and **ONE** part.
2. Attempt any **FIVE FULL** questions.
3. Calculators are allowed.
4. Throughout the paper I is the identity matrix of appropriate order, J is the all ones matrix of appropriate order.

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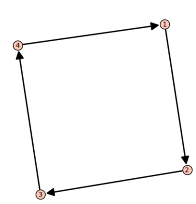
1. (a) Show that the eigenvalues of the adjacency matrix of $K_{p,q}$ are $-\sqrt{pq}$ and $\sqrt{pq}$ with multiplicity 1 and 0 with multiplicity $p+q-2$. [6]

   (b) Let the adjacency matrix of a simple graph G on n vertices be $A(G)$. Prove that the adjacency matrix of its complement is $A(\bar{G}) = J_n - I_n - A(G)$ [4]

2. (a) Let G be a graph with n vertices. Let λ_1 be the largest eigenvalue of $A(G)$, δ be the smallest vertex degree in G, and Δ be the largest vertex degree in G. Show that $\delta \leq \lambda_1 \leq \Delta$. [6]

   (b) Let G be a connected r-regular graph. Prove that r is an eigenvalue of $A(G)$. [4]

3. (a) Let G be the graph drawn below and let Q be its incidence matrix. Show that $Q + Q^T$ is positive semi-definite. [4]



   (b) Let G be a connected graph with n vertices and let M be its incidence matrix. Prove that rank of M is $n-1$ if G is bipartite and n, otherwise. [6]

OR

(c) Let G be a graph with vertex set $\{1, 2, \cdots, n\}$. Let λ_1 be the largest eigenvalue of the Laplacian matrix L. Show that

$$\lambda_1 \leq \max\{d_i + d_j - c(i,j) : 1 \leq i < j \leq n,\ i \sim j\}$$

where $c(i,j)$ is the number of vertices that are adjacent to both i and j. [7]

(d) Compute the eigenvalues of $L(C_n)$, (the Laplacian matrix of cycle graph C_n), where $n \geq 2$. [3]

4. Let T be a tree with $V(T) = \{1, 2, \cdots, n\}$, Laplacian matrix L and algebraic connectivity μ. Let x be a Fiedler vector and n be a characteristic vertex. Let $T_1, T_2, \cdots, T_k$ be the components of $T \setminus \{n\}$. Show that for any $j = 1, 2, \cdots, k$, the vertices of $V(T_j)$ are either all positive, all negative or zero. [10]

5. Let T be a tree. Show that a vertex is a characteristic vertex with respect to a Fiedler vector x if and only if it is characteristic with respect to a Fiedler vector y. [10]

6. (a) Let T be a tree with n vertices. Let D be the distance matrix of T, L the Laplacian matrix and $\tau = [2-d_1, 2-d_2, \cdots, 2-d_n]^T$ where d_i is the degree of vertex i. Show that $LD + 2I = \tau \mathbf{1}^T$. [6]

   (b) Compute the distance matrix of P_5. Can the eigenvalues of the distance matrix of any graph be complex? Justify. [4]

7. (a) Let G be a connected graph with n vertices and $f : E(G) \to \mathbb{R}$ be a unit flow between the vertices i and j. Prove that the resistance distance between i and j is the minimum of $\|f\|^2$. [7]

   (b) Compute the resistance matrix of bipartite graph $K_{1,3}$. [3]