Register number:

Date and session:

ST JOSEPH'S UNIVERSITY, BENGALURU-27
M.Sc (MATHEMATICS) - II SEMESTER
SEMESTER EXAMINATION: April 2024
(Examination conducted in May/June 2024)

MT 8221: MEASURE AND INTEGRATION

(For current batch students only)

Duration: 2 Hours **Max. Marks:** 50

1. The paper contains **TWO** printed pages and **ONE** part.
2. Attempt any **FIVE FULL** questions.

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1. (a) State and prove continuity from below for an arbitrary measure space $(X, \mathcal{S}, \mu)$. **[5]**

   (b) Let A be the subset of $[0, 1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$. **[5]**

2. (a) Let $E \subseteq \mathbb{R}^n$. Show that the following are equivalent:

   i. E is Lebesgue measurable.

   ii. Given an $\varepsilon > 0$, there exists a closed set F_ε such that $F_\varepsilon \subseteq E$ and $\mu_*(E \setminus F_\varepsilon) < \varepsilon$.

   iii. There exists an F_σ set F such that $F \subseteq E$ and $\mu_*(E \setminus F) = 0$. **[7]**

   (b) Show that there exists closed sets A and B with $m(A) = m(B) = 0$, but $m(A + B) > 0$ (Hint: Think of Cantor set) **[3]**

3. (a) Show that if $\{f_n\}$ is a sequence of measurable functions on $(X, \mathcal{S}, \mu)$ then $\sup_n\{f_n(x)\}$ and $\inf_n\{f_n(x)\}$ are also measurable. **[6]**

   (b) Let (X, S) be a measurable space. Show that a function $f : X \to \mathbb{R}$ is measurable if $f^{-1}((r, \infty))$ is a measurable set for every rational number r. **[4]**

   OR

   (c) State and prove Egorov's Theorem. **[10]**

4. (a) State and prove the linearity and additivity properties for simple functions. **[7]**

   (b) Let $f : [0, 1] \to \mathbb{R}$ be measurable and suppose the function $g(x, y) := |f(x) - f(y)|$ is integrable on $[0, 1] \times [0, 1]$. Show that $f(x)$ is integrable on $[0, 1]$. (HINT: Fubini's Theorem). **[3]**

5. (a) State and prove Lebesgue dominated convergence theorem. **[7]**

   (b) If $f = \lim_{n\to\infty} f_n$ then in which of the following case(s) do(es) $\lim_{n\to\infty} \int_X f_n = \int_X f$ hold? **[3]**

   i. $f_n = \chi_{\{1,2,\cdots,n\}}$ on the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N})$, counting measure$)$.

   ii. $f_n = \chi_{\{1,2,\cdots,n\}}$ on the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N})$, Lebesgue measure$)$.

   iii. $f_n = \frac{x}{n}\chi_{[0,1]}$ on the measure space $(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$.

   iv. $f_n = n \cdot \chi_{[0,1/n]}$ on the measure space $(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$.

6. (a) Let p and q be conjugate exponents and let $g \in L^q(X)$. Show that the function $T_g : L^p(X) \to \mathbb{R}$ defined by $T_g(f) = \int_X fg d\mu$ is a continuous linear functional. **[5]**

   (b) Let (X, F, μ) be a measure space. Let $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Let $f \in L^p(\mu)$, $g \in L^q(\mu)$ and $h \in L^r(\mu)$. Show that $fgh \in L^1(\mu)$. (HINT: Hölder's inequality) **[5]**

7. (a) Let $[a, b] \subset \mathbb{R}$ and f be a function of bounded variation on $[a, b]$. Show that f is bounded and $|f|$ is of bounded variation. **[6]**

   (b) Show that if $f : [a, b] \to \mathbb{R}$ is absolutely continuous then f maps measure 0 sets to measure 0 sets. **[4]**