Register Number:

Date:

St. Joseph's College, Autonomous, Bangalore
M.Sc Mathematics-II Semester
End semester Examination: April,2018
MT8114: Algebra-II

Duration: 2.5 Hours **Max. Marks:**70

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1. The paper contains two printed pages.
2. Attempt any **SEVEN FULL** questions.
3. Each question carries 10 marks.
4. **In all questions A is a commutative ring with unity.**

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1. (a) Let $f : A \to B$ be homomorphism of rings. Let J be an ideal of B.
   (i) Prove that $f^{-1}(J)$ is an ideal of A.
   (ii) If J is prime in B, then is $f^{-1}(J)$ prime in A? Justify your answer.
   (iii) If J is maximal in B, then is $f^{-1}(J)$ maximal in A? Justify your answer
   [4+2+1 marks]
   (b) Let I be an ideal of a ring A. Define the radical of $I := r(I) = \{x \in A | x^n \in I \text{ for some } n \in \mathbb{N}\}$. Prove that $r(I)$ is the intersection of all prime ideals of A containing I. [3 marks]

2. (a) State Nakayama's Lemma [2 marks]
   (b) Let Σ be a set partially ordered with respect to the relation " $\leqslant$ ". Prove that the following are equivalent.
   1. Every increasing sequence $x_1 \leqslant x_2 \cdots \leqslant x_n \cdots$ in Σ is stationary.
   2. Every non-empty subset of Σ has a maximal element. [8 marks]

3. (a) Let M', M, M'', N be A-modules.
   Given $u : M' \to M$, we define $\bar{u} : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N)$ as follows: $\bar{u}(f) = f \circ u$ for all $f \in \mathrm{Hom}_A(M, N)$. It can be easily verified that $\bar{u}$ is an A-module homomorphism.
   Let
   $$M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$
   be exact sequence of homomorphism of A-modules. Prove that the following sequence of A-module homomorphisms
   $$0 \to \mathrm{Hom}_A(M'', N) \xrightarrow{\bar{v}} \mathrm{Hom}_A(M, N) \xrightarrow{\bar{u}} \mathrm{Hom}_A(M', N)$$
   is also exact, where $\bar{v}$ is defined similar to $\bar{u}$. [10 marks]
   (b) State Snake's Lemma.

4. State and Prove Hilbert Basis Theorem [10 marks]

5. (a) Prove that in an Artinian ring every prime ideal is maximal. [8 marks]

(b) Give an example of a ring which is neither Noetherian nor Artinian. [2 mark]

6. (a) Suppose that E is an extension of F of prime degree. Show that for $a \in E$ either $F(a) = F$ or $F(a) = E$. [2 marks]

   (b) Prove that $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) = \mathbb{Q}(\sqrt[6]{2})$ [4 marks]

   (c) Let K/F be an extension of fields. Prove that $\alpha \in K$ is algebraic over F if and only if $F(\alpha)/F$ is finite. [4 marks]

7. (a) Prove or Disprove: $\mathbb{Q}(\sqrt[4]{2})$ is Galois over $\mathbb{Q}$. [4 marks]

   (b) Let $\alpha \in \mathbb{Q}$ is a root of a monic polynomial in $\mathbb{Z}[x]$. Prove that α is an integer. [3 marks]

   (c) If ab is algebraic over F ($b \neq 0$), prove that b is algebraic over $F(a)$. [3 marks]

8. Let the extension K/F is Galois, then prove that K is the splitting field of some separable polynomial over F. [10 marks]

9. For each part give an example of a field with stated property. If no such field exists, write ”none”. No justifications are required. [**2 marks each**]

   (a) A field of characteristic 3 which is not finite.

   (b) A finite field of characteristic 0.

   (c) A field of degree 2 over $\mathbb{Q}$ which is not Galois.

   (d) A field of degree 3 over $\mathbb{Q}$ which is not Galois.

   (e) A Galois extension of $\mathbb{F}_3$ whose Galois group is not cyclic.

10. Find the splitting field E of $x^4 + 1$ over $\mathbb{Q}$. Find $\mathrm{Gal}(E/\mathbb{Q})$ and all the subgroups of it. Find the corresponding subfields of E. Is there an automorphism of E whose fixed field is $\mathbb{Q}$? [10 marks]