Register Number:

Date:

St. Joseph's College, Autonomous, Bangalore
M.Sc Mathematics-II Semester
End semester Examination: April,2018
MT8114: Algebra-II

Duration: 2.5 Hours **Max. Marks:**70

1. The paper contains two printed pages.
2. Attempt any **SEVEN FULL** questions.
3. Each question carries 10 marks.
4. **In all the questions A is a commutative ring with unity.**

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1. (a) If I is an ideal of A, the radical of I is defined to be $r(I) = \{x \in A | x^n \in I \text{ for some } n \in \mathbb{N}\}$. Prove that $r(\mathbf{p}^n) = \mathbf{p}$ for all $n > 0$, where $\mathbf{p}$ is a prime ideal of A. [2 marks]

   (b) Prove that M is a finitely generated A-module if and only if M is the quotient of A^n for some $n > 0$ [8 marks]

2. (a) An element m of the A-module M is called torsion element if there exists a non-zero $a \in A$ such that $am = 0$. The set of all torsion elements is denoted by

$$\mathsf{Tor}(M) = \{m \in M : am = 0 \text{ for some non-zero } a \in A\}$$

   Prove that $\mathsf{Tor}(M)$ is a submodule of M if A is an integral domain. [4 marks]

   (b) Let A be a ring and $\mathfrak{R}$ be its nilradical. Show that the following are equivalent.

   1. A has exactly one prime ideal.
   2. Every element of A is either an unit or a nilpotent.
   3. $A/\mathfrak{R}$ is a field.

   [6 marks]

3. (a) Let M be an A-module. Prove that M is a Noetherian A-module if and only if every submodule of M is finitely generated. [8 marks]

   (b) State Snake's Lemma. [2 marks]

4. (a) Let $I_1, I_2, \cdots, I_n$ be ideals of A and let $\mathfrak{p}$ be another prime ideal A such that $\cap_{i=1}^{n} I_i \subseteq \mathfrak{p}$. Then prove that $I_i \subseteq \mathfrak{p}$ for some i. Provide example of three ideals I_1, I_2 and J such that $I_1 \cap I_2 \subseteq J$ but neither $I_1 \subseteq J$ nor $I_2 \subseteq J$. [5 marks]

   (b) Prove that an Artinian ring has only a finite number of maximal ideals. [5 marks]

5. (a) Let $K : F$ be a field extension. An element $\alpha \in K$ is algebraic over F if and only if the simple extension $F(\alpha)/F$ is finite. Deduce that K/F is finite implies K/F is algebraic. Is the converse "Every algebraic extension is finite" true? [5 marks]

(b) Use only straightedge and compass to draw a line segment one-third unit. Please write down the steps you used to draw it. [5 makrs]

6. (a) Prove that if K is algebraic over F and L is algebraic over K, then L is algebraic over F. [5 marks]
   (b) Give an example of a ring which is Noetherian but not Artinian. Justify your answer. [3 marks]
   (c) Suppose that E is an extension of F of prime degree. Show that for $a \in E$ either $F(a) = F$ or $F(a) = E$. [2 marks]

7. (a) Prove that for any field F, if $f(x) \in F[x]$ then there exists an extension K of F which is a splitting field for $f(x)$. [6 marks]
   (b) Find the minimal polynomial for $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$. [4 marks]

8. (a) Prove that the characteristic of a field is 0 or a prime number. [2 marks]
   (b) Define prime subfield of a field. [1 mark]
   (c) Prove that finite field has prime characteristic. [3 marks]
   (d) Hence, further prove that a finite field of characteristic p has p^n elements for some $n \in \mathbb{N}$ [3 marks]
   (e) Give an example of an infinite field of positive characteristic. [1 marks]

9. Let the extension K/F is Galois, then prove that K is the splitting field of some separable polynomial over F. [10 marks]

10. Let $\omega = \cos(\frac{2\pi}{7}) + i\sin(\frac{2\pi}{7})$, (i.e., ω is one of the seventh root of unity), consider the field $\mathbb{Q}(\omega)$. How many subfields does it have and what are they? Draw a lattice diagram [10 marks]