Date:

Registration number:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.Sc. MATHEMATICS - I SEMESTER
SEMESTER EXAMINATION: OCTOBER 2021
(Examination conducted in January-March 2022)

MT 7121 – ALGERBA I

Time: 2 ½ hrs **Max Marks**: 70

This question paper contains two printed pages.

Answer any 7 questions

1. a) Define $GL_n(\mathbb{F})$. If $\mathbb{F}$ is a finite field with q elements, then what is $|GL_n(\mathbb{F})|$? (2)
   b) Describe cycle decomposition algorithm. Find the cycle decomposition of (6)
   $$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 2\,1\,6\,4\,5\,9\,3\,7\,8 \end{pmatrix}.$$
   c) Give examples of subgroups H and K of a group G such that $H \trianglelefteq K \trianglelefteq G$ but H is not a normal subgroup of G, where $\trianglelefteq$ stands for 'is a normal subgroup of'. (2)

2. a) Let G be a group. Show that G acting on itself as $g \cdot a = ag^{-1}, \forall a, g \in G$ is a group action. (3)
   b) Let G be a group acting on a set A. For each fixed $g \in G$, define $\sigma_g: \mathrm{A} \to \mathrm{A}$, $\sigma_g(\mathrm{a}) \coloneqq \mathrm{g} \cdot \mathrm{a}$. Prove that the map from G to S_A defined by $g \to \sigma_g$ is a group homomorphism. (3)
   c) List representatives of all conjugacy classes of S_4. (4)

3. a) State and prove the Class Equation. (6)
   b) Find $\tau^{-1}\sigma\tau$ in S_5 where $\tau = (1\ 2\ 3)(4\ 5)$ and $\sigma = (1\ 4)(2\ 5)$. (2)
   c) Pick out which of the following **CANNOT** be the class equation of the group? (2)
   i) 9 = 1 + 2 + 3 + 3 ii) 3 = 1 + 1 + 1
   iii) 10 = 1 + 2 + 2 + 2 + 3 iv) 6 = 1 + 2 + 3

4. a) Prove that the automorphism group of the cyclic group of order n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$. (8)
   b) Find the $|\mathrm{Aut}(Z_{23})|$. Is $\mathrm{Aut}(Z_{23})$ be cyclic? Justify. (2)

5. a) Show that all Sylow p-subgroups of a group G of order 35 is normal and characteristic in G. Also find the order of Sylow subgroups. (7)
   b) Let $P \in Syl_p(G)$. Prove that P is a unique Sylow p-subgroup iff P is normal. (3)

6. a) State Fundamental Theorem of Finite Abelian Groups. (2)
   b) Let $G = \{1,4,11,14,16,19,26,29,31,34,41,44\}$ under multiplication modulo 45. Express it as a direct product of cyclic groups. (8)

7. a) State Division Algorithm of $F[x]$, where F is a field. (2)

b) Prove that a polynomial of degree n over a field has at most n zeros, counting multiplicity. (6) (2)

c) Find the reminder and quotient on dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$ in $\mathbb{Z}_5[x]$.

8. a) State and prove Gauss's Lemma. (6)

b) Check whether the polynomial $2x^3 + x + 7$ is irreducible over $\mathbb{Q}$. Justify your answer. (2)

c) Find the roots of the polynomial $x^3 - 3x^2 + 2x + 6$ over $\mathbb{Q}$. (2)

9. a) Show that the ideal $\langle x^4 + x^3 + x^2 + x + 1\rangle$ is maximal ideal in $\mathbb{Q}[x]$. (3)

b) Give a field with 9 elements with proper justification. (3)

c) Show that the polynomial $2x^2 - 6$ does not have a zero in $\mathbb{Z}$, but is reducible over $\mathbb{Z}$. Check whether the polynomial is irreducible over $\mathbb{Q}$ and $\mathbb{R}$. (4)

10. a) Define a prime element in an integral domain D. Give an example. (2)

b) Show that $1 + \sqrt{-3} \in \mathbb{Z}[\sqrt{-3}]$ is irreducible. (4)

c) Prove that in an integral domain, every prime is an irreducible. (4)