Date:

Registration number:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
B.Sc., MATHEMATICS - V SEMESTER
SEMESTER EXAMINATION: OCTOBER 2021
(Examination conducted in January-March 2022)

MT 5118 - MATHEMATICS V

Time- 2 ½ hrs Max Marks-70

This question paper contains two printed pages and three parts

I. Answer any 5 questions (5 x 2 =10)

1. Define nilpotent element in a ring. List any two nilpotents of the ring $\mathbb{Z}_8$.
2. Define integral domain and field.
3. Show that $3\mathbb{Z}$ is an ideal of $\mathbb{Z}$.
4. Show that $4\mathbb{Z}$ is neither a prime ideal nor a maximal ideal of $\mathbb{Z}$.
5. Let R be a commutative ring of characteristic 2. Show that the function $f : R \to R$ defined by $f(a) = a^2, \forall a \in R$ is a ring homomorphism.
6. Find the value of a_0 in the Fourier series expansion of $f(x) = e^x$ in $(0,2\pi)$.
7. Define Beta and Gamma function.
8. Prove that $\Gamma(n) = \int_0^\infty 2e^{-x^2}x^{2n-1}\,dx$.

II. Answer any 7 questions (7 x 6 = 42)

9. a. Define idmpotent element in a ring.
   b. In a Boolean ring R, prove that
      i. $a + a = 0, \forall a \in R$.
      ii. $a + b = 0 \Rightarrow a = b, \forall a, b \in R$.
      iii. $ab = ba, \forall a, b \in R$. **[1+5]**
10. a. Define the center of a ring. Show that center of a ring R is a subring of R.
    b. Show that the set $\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}\right\}$ is a subring of $M_2(\mathbb{Z})$. **[4+2]**
11. Define zero divisor in a commutative ring. Prove that a nonzero element $a \in \mathbb{Z}_n$ is either a unit or a zero divisor.
12. a. Define characteristic of a ring. What is the characteristic of $M_2(\mathbb{Z}_3)$.
    b. Let R be a ring with unity 1. Prove that if 1is of infinite order under addition, then characteristic of R is zero, and that if 1is of order n under addition, then characteristic of R is n. **[2+4]**

13. Define left ideal, right ideal and two-sided ideal in a ring. Show that the set $\left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}\right\}$ is a left ideal but not a right ideal of $M_2(\mathbb{Z})$.

14. Define prime ideal. Let R be a commutative ring with unity and A be an ideal of R. Prove that $\frac{R}{A}$ is an integral domain if and only if A is a prime ideal.

15. a. Define maximal ideal. Show that $\{0\}$ is a prime ideal of $\mathbb{Z}$ but not maximal.
    b. Show that for $n > 1$, $n\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ if and only if n is a prime number. **[3+3]**

16. a. Define homomorphism and isomorphism of rings.
    b. Show that the function $\varphi: \mathbb{C} \to \mathbb{C}$ defined by $\varphi(z) = \bar{z}, \forall\, z \in \mathbb{C}$ is a ring isomorphism. **[2+4]**

17. Let R and S be rings and $\varphi: R \to S$ be an onto ring homomorphism. Prove the following:
    i. If R is a ring with unity 1 and $S \neq 0$, then $\varphi(1)$ is the unity of S.
    ii. If A is an ideal of R, show that $\varphi(A)$ is an ideal of S.

III. <u>Answer any 3 questions</u> **(3 x 6 = 18)**

18. Find the half range sine series expansion for $f(x) = \begin{cases} x, & 0 \le x \le \frac{l}{2} \\ l - x, & \frac{l}{2} \le x \le l \end{cases}$.

19. Obtain the Fourier series expansion of $f(x) = x^2$ on $-\pi \le x \le \pi$ with period 2π.

20. Evaluate i) $\beta\left(\frac{5}{2}, \frac{7}{2}\right)$ ii) $\int_0^{\pi/2} \sin^6\theta \, d\theta$. **[2+4]**

21. Prove that i) $n\Gamma(n) = \int_0^\infty e^{(-x)^{1/n}} dx$ ii) $\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx$. **[3+3]**

22. Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.